

A differential operator for integrating one-loop scattering equations

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ABSTRACT: We propose a differential operator for computing the residues associated with a class of meromorphic n -forms that frequently appear in the Cachazo-He-Yuan form of the scattering amplitudes. This differential operator is conjectured to be uniquely determined by the local duality theorem and the intersection number of the divisors in the n -form. We use the operator to evaluate the tree-level amplitude of ϕ^3 theory and the one-loop integrand of Yang-Mills theory from their CHY forms. The method can reduce the complexity of the calculation. In addition, the expression for the 1-loop four-point Yang-Mills integrand obtained in our approach has a clear correspondence with the Q-cut results.

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1 Introduction

Cachazo, He and Yuan discovered a succinct form of writing the scattering amplitudes of various (quantum) field theories [1–3]. Not only does it make many duality properties manifest, e.g. the KLT relation [4] between gauge theory and gravity [5] as well as the KK relation [6] and the BCJ relation [7], but it is also a great platform for constructing new theories, with or without Lagrangian descriptions [8]. A flurry of activities ensued.

In four dimensions, tree-level scattering equations can be viewed as the constraints of the Roiban-Spradlin-Volovich (RSV) formula in $\mathcal{N} = 4$ super Yang-Mills [9, 10]. The $\mathcal{N} = 8$ supergravity tree amplitudes, proposed and derived from twistor string theory [11, 12], can also be included in this framework [13–15]. In d dimensions, the polynomial form of the scattering equations is first obtained in [16]. The algebraic varieties corresponding to these homogeneous polynomials are studied by [17].

A few solutions of the tree-level scattering equations in four dimensions in some special kinematic limits are obtained: tree level scattering amplitudes for n gluons and n gravitons are computed in [18], the tree-level scattering equations are solved up to six-points [19], while a general relation between solutions of the tree-level scattering equations is proposed, and checked at special kinematic limits up to six points [20].

MHV tree-level amplitudes of gravity and gluons are obtained from the CHY amplitudes in [21]. A five-point scattering amplitude in Yang-Mills theory is obtained in [22] by exploiting the Vieta formula (relating the sum of the solutions of a polynomial equation to the coefficients of the polynomial). The elimination theory is applied to reduce the polynomials and obtained the residues of the scattering equations in [23–25].

Methods based on algebraic geometry are exploited to improve the efficiency of computations. Companion matrix method [26] and Bezoutian matrix method [27] are used to evaluate the CHY expressions, without solving the scattering equations, and checked against the 5-point amplitude in ϕ^3 theory analytically, as well as higher point amplitudes numerically. In [28] polynomial reduction techniques are used to cast the scattering equations into the standard basis, called the H-basis. In [29] another prescription for such evaluation is proposed, using the polynomial reduction techniques,

and explicit results are given analytically up to 3-points at one loop in ϕ^3 theory. These approaches do not depend on the particular theory and the method we are going to propose also belongs to this category.

Building blocks method was proposed by [30] in which higher-point CHY-integrands are reduced a la BCFW to basic building blocks. It was then further developed in [31] to a systematic Λ -algorithm, and then used in [32] to propose and compute a 1-loop CHY amplitude for the n -gon. Integration rules for higher-point amplitudes are derived in [33–38] to facilitate practical computations. High order poles were discussed in [39, 40].

In [41] one loop scattering amplitudes are obtained from tree-level ones in one higher spatial dimension. A universal all-loop CHY form was constructed from tree-level CHY forms in ϕ^3 theory in [42] and checked against Q-cut results.

These methods have various degrees of success in constructing higher-loop amplitudes in scalar theories as successful methods have been developed to subtract the forward singularities arisen in gluing tree-level amplitudes to form one-loop amplitudes. These methods can hence be generalized in a straightforward way to the computations of scattering amplitudes in gauge theory and gravity at tree-level; but so far no general method is in sight for removing the forward singularities introduced when gluing tree amplitudes to form loop amplitudes in gauge or gravity theories.

On the other hand the scattering equations at loop levels are derived from ambitwistor string theory [43–45]. The CHY expressions are then extended to one and two loops for the bi-adjoint scalar, gauge theory, and gravity in [46–48].

In this paper, we propose a new method, also based on algebraic geometry, to evaluate CHY forms. A differential operator, constructed in a systematic procedure, is conjectured to capture the residue around the contour associated with the scattering equations. A one-to-one correspondence to the Q-cut results for the 1-loop four-point CHY expression in SYM is made manifest by our algorithm. Compared with other algebraic geometry based techniques, we avoid the task of finding the Gröbner basis [49] of the scattering equations. The construction of this operator demands information mostly from the integration contour; and therefore the complexity of the theory-specific factors in CHY forms has little impact on the procedure. The residues at the phantom poles resulted from factorization in the polynomial form of scattering equations naturally vanish in our method.

2 A differential operator for multivariable residues

The CHY-form provides a beautiful and compact expression for scattering amplitudes. Schematically, an n -particle scattering process reads

$$\int \frac{d\sigma_1 \cdots d\sigma_n}{\text{vol}(\text{Residual Symmetry})} \delta(\text{Scattering Equations}) \mathcal{I}, \quad (2.1)$$

with σ_i 's being complex variables which can, in turn, be related to the worldsheet coordinates of the vertex operators in string theory. The integrand \mathcal{I} depends on the underlying theory. These scattering equations are typically a set of rational equations in σ_i 's, whose coefficients encoding the dynamic information of the scattering process.

In the language of complex analysis, integrals with delta-functions are equivalent to residues around the contours defined by these delta-function constraints. The CHY-form (2.1) can be casted into a residue associated with a meromorphic form as follows,

$$\oint \frac{d\sigma_1 \wedge \cdots \wedge d\sigma_{n-m}}{h_1 \cdots h_{n-m}} \mathcal{I}', \quad (2.2)$$

where m is the number of the residual symmetry generators and h_i 's are polynomials originated from the scattering equations. These polynomials are, roughly speaking, the numerators in the polynomial scattering equations.

In this section, we introduce a differential operator for computing such residues.

Conjecture 2.1. : *Given a polynomial ideal $\langle f_1, f_2 \cdots f_k \rangle$ in variables $z_1, z_2 \cdots z_k$, the polynomials are homogeneous and their degrees are d_1, d_2, \cdots, d_k respectively. If the solution to the corresponding algebraic equations is an isolated point p , the residue associated with a meromorphic form has a differential interpretation. Namely, for a holomorphic function $\mathcal{R}(z_i)$ in the neighbourhood of the point p ,*

$$\text{Res}_{\{(f_1), \dots, (f_k)\}, p}[\mathcal{R}] \equiv \oint \frac{dz_1 \wedge \cdots \wedge dz_k}{f_1 \cdots f_k} \mathcal{R} = \mathbb{D}[\mathcal{R}], \quad \mathbb{D} = \sum_{\{s_i\}} a_{s_1 s_2 \cdots s_k} \partial_{r_1}^{s_1} \partial_{r_2}^{s_2} \cdots \partial_{r_k}^{s_k}, \quad (2.3)$$

where the coefficients $a_{s_1 s_2 \cdots s_k}$ are constants independent of z_i and $\partial_{r_i}^{s_i} = \frac{\partial^{s_i}}{\partial z_{r_i}^{s_i}}, i \in [1, k]$. The summation is taken over all the solutions to the Frobenius equation:

$$\sum_{i=1}^k s_i = \sum_{h=1}^k d_h - k. \quad (2.4)$$

Furthermore, the differential operator \mathbb{D} is uniquely determined by requiring the residue to satisfy the local duality theorem [50, 51] and to give the correct intersection number of the divisors $D_i = (f_i)$.¹

We shall exploit this conjecture to evaluate CHY scattering equation (2.2). Typically, the polynomials h_i in (2.2) are not homogeneous, and therefore include extra poles that are not solutions to the original scattering equations. We call these “spurious poles,” and the locations of them are easy to determine. In Section 4, we shall introduce a “homogenization” procedure to case these polynomials and the resulting integrals to meet the conditions of the conjecture.

The differential operator \mathbb{D} computes the sum of residues around all the solutions of $h_1 = \cdots = h_{n-m} = 0$, including the spurious ones. Therefore it is crucial to demonstrate how to remove the contributions from the spurious poles. A calculation of CHY amplitudes can sometimes involve computing the residues at infinity. This is achieved using our conjecture, as demonstrated in Section 4. By computing residues at the finite poles, the spurious poles and the poles at infinity, scattering amplitude can be conveniently evaluated from the CHY forms.

3 A Warm-up example

In this section we study the 5-point tree-level amplitude in the massless ϕ^3 theory and use it as a toy model to illustrate the evaluation of the multi-variable contour integrals that often appear in CHY-form by the proposed operator \mathbb{D} . We denote this amplitude as \mathcal{A}_{ϕ^3} and its explicit expression is given in [16, 26, 27],

$$\mathcal{A}_{\phi^3} = \oint_{h_1=h_2=0} \frac{d\sigma_1 \wedge d\sigma_2}{h_1 h_2 (\sigma_1 - \sigma_2)} + \frac{d\sigma_1 \wedge d\sigma_2}{h_1 h_2 (1 - \sigma_1) \sigma_2}, \quad (3.1)$$

¹We remark on the motivation for this conjecture. Let us consider the integral,

$$\oint_{f_1=\cdots=f_k=0} \frac{g(z_1, \cdots, z_k) dz_1 \wedge \cdots \wedge dz_k}{f_1 \cdots f_k},$$

where f_i ’s are homogeneous polynomials in z_i ’s and the numerator g is a monomial in z_i of degree M . The residue is non-vanishing if and only if $M = \sum_{i=1}^k d_{f_i}$ [27]. Therefore, the differential operator that computes the residue can involve only derivatives of degree M . This observation generalizes naturally to the case in which the numerator is a polynomial.

where h_1, h_2 being polynomials in σ_1 and σ_2 , are roots of the tree-level scattering equations. Their solutions take the following forms,

$$h_1 = k_{13}\sigma_1 + k_{14}\sigma_2 + k_{12}, \quad (3.2)$$

$$h_2 = k_{45}\sigma_1 + k_{25}\sigma_1\sigma_2 + k_{35}\sigma_2, \quad (3.3)$$

where $k_{ij} = \frac{1}{2}(k_i + k_j)^2$.

The two terms in (3.1) can be integrated separately. In the first term we denote the remaining factor in the denominator as $h_0 = (\sigma_1 - \sigma_2)$. These three polynomials h_1, h_2, h_0 can be made homogeneous:

$$\begin{aligned} \tilde{h}_0 &= (\sigma_1 - \sigma_2), \\ \tilde{h}_1 &= k_{13}\sigma_1 + k_{14}\sigma_2 + k_{12}\sigma_0, \\ \tilde{h}_2 &= k_{45}\sigma_0\sigma_1 + k_{25}\sigma_1\sigma_2 + k_{35}\sigma_0\sigma_2, \end{aligned}$$

by introducing an extra variable σ_0 which will be later integrated out. Hence the first contour integral becomes,

$$\begin{aligned} \oint_{h_1=h_2=0} \frac{d\sigma_1 \wedge d\sigma_2}{h_1 h_2 (\sigma_1 - \sigma_2)} &= \oint_{\tilde{h}_1=\tilde{h}_2=\sigma_0-1=0} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_0}{\tilde{h}_1 \tilde{h}_2 (\sigma_0 - 1)} \frac{1}{\tilde{h}_0} \\ &= - \oint_{\tilde{h}_1=\tilde{h}_2=\tilde{h}_0=0} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_0}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_0} \frac{1}{(\sigma_0 - 1)}, \end{aligned} \quad (3.4)$$

where in the second equality sign we have used the global residue theorem and the intersecting divisors are all generated by homogeneous polynomials. This contour integral is immediately computed by our conjecture. The corresponding differential operator takes the form,

$$\mathbb{D} = a_{100} \frac{\partial}{\partial \sigma_1} + a_{010} \frac{\partial}{\partial \sigma_2} + a_{001} \frac{\partial}{\partial \sigma_0}. \quad (3.5)$$

The computation of the integral is now translated to finding the values of the coefficients a_{100} , a_{010} and a_{001} . The local duality theorem requires,

$$\oint_{\tilde{h}_1=\tilde{h}_2=\tilde{h}_0=0} \frac{\tilde{h}_1 d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_0}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_0} = 0, \quad \oint_{\tilde{h}_1=\tilde{h}_2=\tilde{h}_0=0} \frac{\tilde{h}_0 d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_0}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_0} = 0. \quad (3.6)$$

The intersection number of the divisors here is 2 and this yields,

$$\oint_{\tilde{h}_1=\tilde{h}_2=\tilde{h}_0=0} \frac{d\tilde{h}_1 \wedge d\tilde{h}_2 \wedge d\tilde{h}_0}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_0} = \oint_{\tilde{h}_1=\tilde{h}_2=\tilde{h}_0=0} \frac{\mathcal{J} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_0}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_0} = 2, \quad (3.7)$$

where $\mathcal{J} = \det(\frac{\partial \tilde{h}_i}{\partial \sigma_j})$ is the Jacobian of integral parameter transformation. In the language of the differential operator, conditions (3.6) and (3.7) read,

$$\mathbb{D}\tilde{h}_1 = 0, \quad \mathbb{D}\tilde{h}_0 = 0, \quad \mathbb{D}\mathcal{J} = 2. \quad (3.8)$$

Solving for a 's the constraints above, we obtain,

$$a_{100} = -\frac{2k_{12}}{G_1}, \quad a_{010} = -\frac{2k_{12}}{G_1}, \quad a_{001} = -\frac{2(k_{13} + k_{14})}{G_1}.$$

where

$$G_1 = -2k_{12}^2 k_{134} + 2k_{13}k_{12}k_{123} + 2k_{14}k_{12}k_{123} + 2k_{13}k_{12}k_{124} + 2k_{14}k_{12}k_{124}.$$

Thus the action of the differential operator gives,

$$\oint_{\tilde{h}_0=\tilde{h}_2=\tilde{h}_0=0} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_0}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_0} \frac{1}{(\sigma_0 - 1)} = \mathbb{D} \left(\frac{1}{\sigma_0 - 1} \right) = -\frac{a_{001}}{(\sigma_0 - 1)^2} \Big|_{\sigma_0 \rightarrow 0} = \frac{2(k_{13} + k_{14})}{G_1}. \quad (3.9)$$

Similarly, the second term in (3.1) can also be related to another integral in which the intersecting divisors are originated from homogeneous polynomials only. The residue of the latter is then represented by a second order differential operator \mathbb{D} ,

$$\mathbb{D} = a_{002} \frac{\partial^2}{\partial \sigma_0^2} + a_{011} \frac{\partial}{\partial \sigma_0} \frac{\partial}{\partial \sigma_2} + a_{020} \frac{\partial^2}{\partial \sigma_2^2} + a_{101} \frac{\partial}{\partial \sigma_1} \frac{\partial}{\partial \sigma_0} + a_{110} \frac{\partial}{\partial \sigma_1} \frac{\partial}{\partial \sigma_2} + a_{200} \frac{\partial^2}{\partial \sigma_1^2}.$$

The residue computed by this operator is,

$$\mathbb{D} \left(\frac{1}{\sigma_0 - 1} \right) = -\frac{32k_{13}k_{14}(k_{13}(k_{124} + k_{134}) - k_{14}k_{123})}{G_2},$$

$$G_2 = 32k_{12}k_{13}k_{14}k_{123}(-k_{14}k_{123} + k_{12}k_{124} + k_{13}k_{124} + k_{12}k_{134} + k_{13}k_{134}).$$

Putting together the two terms, we obtain,

$$\mathcal{A}_{\phi^3} = -\frac{2(k_{13} + k_{14})}{G_1} + \frac{32k_{13}k_{14}(k_{13}(k_{124} + k_{134}) - k_{14}k_{123})}{G_2}. \quad (3.10)$$

The expression is identical with those in [16, 26, 27].

4 Direct evaluation of one-loop CHY-form

In this section we provide a systematic algorithm that exploits the conjecture (2.1) in the calculation of CHY forms.

Before discussing the technical details, let us first sketch out the steps in words. The polynomial form of scattering equations is our starting point. The general transformations from the standard scattering equations to the polynomial ones are given in [16]. These transformations introduce a Jacobian into the CHY expression that is simply the Vandermonde determinant. The polynomial equations are not entirely equivalent to the original equations, rather, they bring in extra solutions that do not satisfy the original.

As explained in Section 2, the evaluation of a CHY-form boils down to computing the residue (2.2). Since the polynomial equations have extra solutions, (2.2) can be obtained by computing the sum of the residues at *all* the poles of said polynomials first and then removing the contribution from the extra poles. To calculate these residues directly can be quite demanding and this is where Conjecture 2.1 comes in handy.

Depending on the specific expression of the CHY-form, we may encounter two kinds of meromorphic forms: one that is regular at infinity and one that has non-vanishing residues at infinity. To compute the total residue in the former category, we adopt a straightforward procedure of homogenizing the polynomials such that all the poles are condensed at one single isolated point. Hence the total residue can be immediately determined by our conjecture. The latter category can be attacked in a similar fashion, after we embed the complex manifold on which the corresponding meromorphic form lives into a compact one and make a point at infinity well-defined. This process will be discussed in detail.

As for the residues at the phantom poles, we observe that these poles are trivial to locate, however, for reasons that will become clear later, the aforementioned homogenization does not work for this case. A modified conjecture will be given for computing such residues.

4.1 Polynomial scattering equations

An n -point scattering amplitude in D dimensions takes the form,

$$\mathcal{A}_n^{l=1} = \int \frac{d^D \ell}{\ell^2} \mathcal{I}_n^{l=1}(\text{kinematics}), \quad (4.1)$$

where ℓ denotes the loop momentum and the exact expression for the integrand $\mathcal{I}_n^{l=1}$ depends on the underlying quantum field theory. As shown in [46–48], in a variety of

theories, the integrand has a CHY representation that schematically reads,

$$\mathcal{I}_n^{l=1} = \oint_{f_1=\dots=f_{n-1}=0} \frac{d\sigma_1 \wedge \dots \wedge d\sigma_{n-1}}{f_1 \dots f_{n-1}} \frac{\mathcal{N}(\sigma_i)}{\mathcal{D}(\sigma_i)}, \quad (4.2)$$

where $\mathcal{N}(\sigma_i)$ and $\mathcal{D}(\sigma_i)$ are polynomials in σ_i 's and encode the kinematic information of the scattering amplitude. The rational function f_i denotes the i -th one-loop scattering equation, originally derived in the context of the high-energy limit of string theory in [52], and later re-discovered in ambitwistor string in [43, 44],

$$f_i = \frac{\ell \cdot k_i}{\sigma_i} + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0. \quad (4.3)$$

While these equations are difficult to solve analytically, Dolan *et al* introduced in [16] the following transformations to make them more friendly,

$$g_m = \sum_{i=1}^n \sigma_i^{m+1} f_i, \quad m \in \{-1, 0, \dots\}. \quad (4.4)$$

By counting the degrees of freedom, obviously only $n - 1$ equations are independent. We choose the $(n - 1)$ ones with the lowest degrees, that is, $g_i = 0$ ($i = 1, \dots, n - 1$).² The transformations from $\{f_1, \dots, f_{n-1}\}$ to $\{g_1, \dots, g_{n-1}\}$ bring into the integrand the following Jacobian that can be easily computed [16],

$$\begin{aligned} \mathcal{J} &= \det \begin{pmatrix} \sigma_1(\sigma_1 - \sigma_n) & \sigma_2(\sigma_2 - \sigma_n) & \dots & \sigma_{n-1}(\sigma_{n-1} - \sigma_n) \\ \sigma_1(\sigma_1^2 - \sigma_n^2) & \sigma_2(\sigma_2^2 - \sigma_n^2) & \dots & \sigma_{n-1}(\sigma_{n-1}^2 - \sigma_n^2) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1(\sigma_1^{n-1} - \sigma_n^{n-1}) & \sigma_2(\sigma_2^{n-1} - \sigma_n^{n-1}) & \dots & \sigma_{n-1}(\sigma_{n-1}^{n-1} - \sigma_n^{n-1}) \end{pmatrix} \\ &= \sigma_1 \sigma_2 \dots \sigma_{n-1} \prod_{1 < i < j < n} (\sigma_i - \sigma_j). \end{aligned} \quad (4.5)$$

The explicit expressions for these new equations read,

$$g_m = \sum_{i=1}^n p \cdot k_i \sigma_i^m + \sum_{i < j}^n k_i \cdot k_j \left(\sum_{r=1}^{m-1} \sigma_i^r \sigma_j^{m-r} \right) = 0, \quad m = 1, 2, \dots, n - 1. \quad (4.6)$$

These polynomial equations already provide us a much better platform than f_i since each of them is of degree m while all the f_i 's give rise to equations of degree $n - 2$ (with

² g_{-1} is still a rational function and $g_0 = 0$ is satisfied trivially by the conservation of momentum.

the choice of gauge $\sigma_n = 1$). We can further simplify the equations by applying a few more linear transformations given as follows,

$$\begin{aligned}
h_1 &= g_1, \\
h_2 &= g_2 - g_1 \left(\sum_i^n \sigma_i \right), \\
h_3 &= g_3 - g_2 \left(\sum_i^n \sigma_i \right) + \frac{g_1}{2} \left(\sum_{i \neq j}^n \sigma_i \sigma_j \right), \\
&\dots \\
h_{n-1} &= g_{n-1} - g_{n-2} \left(\sum_i^n \sigma_i \right) + \frac{g_{n-3}}{2} \left(\sum_{i_1 \neq i_2}^n \sigma_{i_1} \sigma_{i_2} \right) + \dots \\
&\quad + \frac{g_{n-1-m}}{(-)^m m!} \left(\sum_{i_1 \neq \dots \neq i_m}^n \prod_{r=1}^m \sigma_{i_r} \right) + \dots + \frac{(-)^{n-2} g_1}{(n-2)!} \left(\sum_{i_1 \neq \dots \neq i_{n-2}}^n \prod_{r=1}^{n-2} \sigma_{i_r} \right). \quad (4.7)
\end{aligned}$$

The Jacobian for the additional transformations above is simply 1. Explicitly, we write down the h_i 's,

$$\begin{aligned}
h_1 &= \sum_{i=1}^n l_i \sigma_i \\
h_m &= (-)^{m-1} \sum_{i_1 < i_2 < \dots < i_m}^n \sigma_{i_1 \dots i_m} l_{i_1 \dots i_m} \quad (4.8)
\end{aligned}$$

where $m \in \{2, \dots, n-1\}$ and we have used the following notations,

$$\sigma_{i_1 \dots i_m} \equiv \prod_{r=1}^m \sigma_{i_r}, \quad l_i \equiv l \cdot k_i, \quad l_{i_1 \dots i_m} \equiv \left(l \cdot k_{i_1 \dots i_m} - \frac{1}{2} k_{i_1 \dots i_m}^2 \right), \quad k_{i_1 \dots i_m} \equiv \sum_{r=1}^m k_{i_r}.$$

As mentioned before, the polynomial equations $h_i = 0$ ($i = 1, \dots, n-1$) have more solutions than the original scattering equations. These extra solutions locate at $\sigma_1 = \sigma_2 = \dots = \sigma_{n-1} = 1$ in the gauge $\sigma_n = 1$.

Now the one-loop CHY-form (4.2) can be rewritten as the following

$$\begin{aligned}
& \mathcal{I}_n^{l=1} \\
&= \oint_{h_1=\dots=h_{n-1}=0} \frac{d\sigma_1 \wedge \dots \wedge d\sigma_{n-1}}{h_1 \dots h_{n-1}} \frac{\mathcal{N}'(\sigma_i)}{\mathcal{D}(\sigma_i)} - \oint_{\sigma_1=\dots=\sigma_{n-1}=1} \frac{d\sigma_1 \wedge \dots \wedge d\sigma_{n-1}}{h_1 \dots h_{n-1}} \frac{\mathcal{N}'(\sigma_i)}{\mathcal{D}(\sigma_i)} \\
&= \oint_{h_1=\dots=h_{n-1}=0} \frac{d\sigma_1 \wedge \dots \wedge d\sigma_{n-1}}{h_1 \dots h_{n-1}} \frac{\mathcal{N}_{reg}(\sigma_i)}{\mathcal{D}(\sigma_i)} + \oint_{h_1=\dots=h_{n-1}=0} \frac{d\sigma_1 \wedge \dots \wedge d\sigma_{n-1}}{h_1 \dots h_{n-1}} \frac{\mathcal{N}_\infty(\sigma_i)}{\mathcal{D}(\sigma_i)} \\
&\quad - \oint_{\sigma_1=\dots=\sigma_{n-1}=1} \frac{d\sigma_1 \wedge \dots \wedge d\sigma_{n-1}}{h_1 \dots h_{n-1}} \frac{\mathcal{N}'(\sigma_i)}{\mathcal{D}(\sigma_i)}, \tag{4.9}
\end{aligned}$$

where $\mathcal{N}'(\sigma_i) = \mathcal{J}(\sigma_i)\mathcal{N}(\sigma_i) = \mathcal{N}_{reg} + \mathcal{N}_\infty$. In the second equal sign we have separated the numerator \mathcal{N}' into two parts, with

$$\deg(\mathcal{N}_{reg}) < \deg(h_1) + \dots \deg(h_{n-1}) + \deg(\mathcal{D}) - (n-1), \tag{4.10}$$

$$\deg(\mathcal{N}_\infty) \geq \deg(h_1) + \dots \deg(h_{n-1}) + \deg(\mathcal{D}) - (n-1). \tag{4.11}$$

The integrand containing \mathcal{N}_{reg} has no residues at infinity while the one containing \mathcal{N}_∞ does. In the rest of this section, we calculate these three terms in (4.9), using our conjecture.

4.2 Residues at finite poles

In this subsection, we demonstrate how to utilize Conjecture 2.1 in computing the first term in the second equal sign of (4.9). This term is equal to the sum of residues associated with the n -form regular at infinity:

$$\Omega = \frac{d\sigma_1 \wedge \dots \wedge d\sigma_{n-1}}{h_1 \dots h_{n-1}} \frac{\mathcal{N}_{reg}(\sigma_i)}{\mathcal{D}(\sigma_i)}. \tag{4.12}$$

This residue can not be evaluated by the conjecture yet, but will be after the so-called “homogenization” procedure.

Consider one of the polynomials h_i that is of degree d and reads,

$$h_i = \sum_{\{s\}} \alpha_{s_1 \dots s_{n-1}} \sigma_1^{s_1} \dots \sigma_{n-1}^{s_{n-1}}, \tag{4.13}$$

where α ’s are constants of which the explicit expressions are not important and we have $0 \leq s_i \leq d$ for $i = 1, \dots, n-1$ and $s_1 + \dots + s_{n-1} \leq d$ for every term in the summation.

We introduce an additional variable σ_0 and define the homogenization of h_i as the following,

$$\tilde{h}_i = \sum_{\{s\}} \alpha_{s_1 \dots s_{n-1}} \sigma_1^{s_1} \dots \sigma_{n-1}^{s_{n-1}} \sigma_0^{d-s_1-\dots-s_{n-1}}. \quad (4.14)$$

When $\sigma_0 = 1$ we recover the original polynomial $\tilde{h}_i = h_i$. We homogenize all the polynomials h_i , ($i = 1, \dots, n-1$) as well as $h_0 = \mathcal{D}(\sigma_i)$. We construct a new meromorphic form from these homogenized polynomials as follows,

$$\tilde{\Omega} = \frac{\mathcal{N}_{reg}(\sigma_i)}{\tilde{h}_1 \dots \tilde{h}_{n-1} \tilde{h}_0 (\sigma_0 - 1)} d\sigma_1 \wedge \dots \wedge d\sigma_{n-1} \wedge d\sigma_0. \quad (4.15)$$

Since the function in $\tilde{\Omega}$ is regular at infinity, the global residue theorem leads to,

$$0 = \sum_p \text{Res}_{\{D_1, \dots, D_{n-1}, \sigma_0-1\}, p} \tilde{\Omega} + \text{Res}_{\{D_1, \dots, D_{n-1}, D_0\}, p'} \tilde{\Omega}, \quad (4.16)$$

where the divisors are $D_i = (\tilde{h}_i)$, ($i = 0, 1, \dots, n-1$). The first term above recovers the original residues associated with Ω while the second term contains divisors generated solely by homogeneous polynomials. These divisors intersect at only an isolated point that is the origin. Conjecture 2.1 now applies to the second term straightaway.

According to our conjecture, the following differential operator \mathbb{D} fully characterizes the relevant local information of the residue

$$\mathbb{D} = \sum_{\{s_i\}} a_{s_0 \dots s_{n-1}} \partial_0^{s_0} \dots \partial_{n-1}^{s_{n-1}}, \quad (4.17)$$

where the summation is taken over all the non-negative solutions to the Frobenius equation $s_0 + \dots + s_{n-1} = \text{ord}(\mathbb{D})$ and

$$\text{ord}(\mathbb{D}) = \deg(\tilde{h}_1) + \dots + \deg(\tilde{h}_{n-1}) + \deg(\tilde{h}_n) - n. \quad (4.18)$$

The coefficients $a_{s_0 \dots s_{n-1}}$ in the operator \mathbb{D} are uniquely fixed by the local duality theorem and the intersection number of the divisors D_i 's.

The local duality theorem yields,

$$0 = \oint_{\tilde{h}_0 = \tilde{h}_1 = \dots = \tilde{h}_{n-1} = 0} \frac{P Q d\sigma_0 \wedge d\sigma_1 \wedge \dots \wedge d\sigma_{n-1}}{\tilde{h}_0 \dots \tilde{h}_{n-1}}, \quad (4.19)$$

where Q is a polynomial of degree d_Q in the ideal $\langle \tilde{h}_0, \tilde{h}_1, \dots, \tilde{h}_{n-1} \rangle$ and P is a holomorphic function in the neighbourhood of the intersecting point. To extract enough

constraints from (4.19), we need at most have Q run over the homogeneous polynomials \tilde{h}_i and P all the monomials of degree $M - d_Q$ for each Q . The residue of PQ vanishes in each case. Namely the constraints read,

$$\mathbb{D}\left(\tilde{h}_j \prod_{i=0}^{n-1} \sigma_i^{r_i}\right) = 0, \quad j = 0, \dots, n-1, \quad (4.20)$$

for all possible solutions to the equation $\sum_{i=0}^{n-1} r_i = M - \deg(\tilde{h}_j)$ where all r_i 's are non-negative integers. These constraints are not completely independent, in fact they have a large redundancy. By probing a large number of non-trivial examples, we observed that these equations are enough to fix the coefficients $a_{s_0 \dots s_{n-1}}$ up to a global scalar factor.³

We only need one more inhomogeneous equation to determine this factor and the intersection number defined as follows is a natural choice,

$$\prod_{i=0}^{n-1} \deg(\tilde{h}_i) = \oint_{\tilde{h}_0 = \dots = \tilde{h}_{n-1} = 0} \frac{d\tilde{h}_0 \wedge \dots \wedge d\tilde{h}_{n-1}}{\tilde{h}_0 \dots \tilde{h}_{n-1}}. \quad (4.21)$$

This translates to the inhomogeneous constraint,

$$\mathbb{D}\left[\det(\partial_i \tilde{h}_j)\right] = \prod_{i=0}^{n-1} \deg(\tilde{h}_i). \quad (4.22)$$

Combining the independent equations from (4.20) and (4.22), we have just enough constraints to determine the coefficients $a_{s_0 \dots s_{n-1}}$. Therefore we have obtained the value of the residue at the origin, i.e

$$\text{Res}_{\{D_1, \dots, D_{n-1}, D_n\}, p'} \tilde{\Omega} = \mathbb{D}\left[\frac{\mathcal{N}_{reg}(\sigma_i)}{\sigma_0 - 1}\right]. \quad (4.23)$$

4.3 Residues at infinity

Now we move on to the second term in (4.9). Due to the degree of \mathcal{N}_∞ , we need to deal with the poles at infinity.

³We have not yet found a simple way to show that the rank of the homogeneous constraints is exactly $(\text{the number of } a_{s_0 \dots s_{n-1}}) - 1$. We have observed this property in all the examples considered and believe this is true in general. In principle, if the homogeneous constraints are constructed with the generators of \mathcal{O}/I (\mathcal{O} is the polynomial ring of the ideal $I = \langle \tilde{h}_0, \dots, \tilde{h}_{n-1} \rangle$), these constraints are of less redundancy. This property may help in determining the rank of the homogenous constraints.

According to the global residue theorem, the residues of a meromorphic n -form on a compact n -dimensional complex manifold sum up to zero. This gives an equation relating residues at different poles (of the form) on the manifold, and thus provides an alternative method to compute residues: if we are interested in the sum of residues at certain poles, we can compute that of all the other poles on the manifold instead.

Since we are dealing with forms on \mathbb{C}^n which is non-compact, to use the method above, we embed \mathbb{C}^n into a compact n -dim manifold and use the global residue theorem there. A natural choice for the compact manifold is \mathbb{CP}^n , where \mathbb{C}^n can be identified with one of the standard coordinate patches, say $U_0 = \{[z_i]_{i=0,\dots,n} \in \mathbb{CP}^n | z_0 \neq 0\}$, of \mathbb{CP}^n . Then the point(s) at infinity are simply those in the complement of U_0 , i.e. $\{[z_i]_{i=0,\dots,n} \in \mathbb{CP}^n | z_0 = 0\}$. In order to use the global residue theorem, we should not only extend the manifold from \mathbb{C}^n to \mathbb{CP}^n , but also extend the original differential form to the whole \mathbb{CP}^n . Namely, we now regard the original differential form as a *local* expression on the coordinate patch U_0 , and extend⁴ it naturally to \mathbb{CP}^n by the homogenous coordinates. Depending on the original differential form, the extended form may develop poles at infinity, and in that case the global residue theorem simply says that the sum of residues of finite poles and that of poles at infinity is zero. Note that now those points at infinity is in fact no different from those in \mathbb{C}^n , they are at infinity only w.r.t the patch U_0 . In summary, to compute the residue of a pole at infinity we need go to a patch covering that point and then compute as usual.

However, the discussion above is usually not convenient in practice. Here we introduce another method. Suppose we want to compute the sum of residues of all finite poles of the differential form,

$$\Omega = \frac{1}{h_1 \dots h_{n-1}} \frac{\mathcal{N}_\infty}{\mathcal{D}} d\sigma_1 \wedge \dots \wedge d\sigma_{n-1}. \quad (4.24)$$

Then we can consider the global residue theorem for the form

$$\Gamma = \frac{\mathcal{N}_\infty}{\tilde{h}_1 \dots \tilde{h}_{n-1} \tilde{h}_0} d\sigma_1 \wedge \dots \wedge d\sigma_{n-1} \wedge d\sigma_0, \quad (4.25)$$

where $h_0 := \tilde{\mathcal{D}}\sigma_0^m(\sigma_0 - 1)$, \tilde{h}_i and $\tilde{\mathcal{D}}$ mean the homogenized version of h_i and \mathcal{D} as in (4.14) and m is a positive integer such that the following equation is valid

$$\deg(\mathcal{N}_\infty) = \deg(\tilde{h}_1) + \deg(\tilde{h}_2) + \dots + \deg(\tilde{h}_{n-1}) + \deg(h_0) - (n+1). \quad (4.26)$$

⁴which will become clear in later example.

With these choices, the form Γ has no pole at infinity and the sum of all finite residues simply vanishes. Now the set of poles of Γ consists of two parts

$$S_0 = \{\tilde{h}_1 = \dots = \tilde{h}_{n-1} = 0, \tilde{\mathcal{D}}\sigma_0^m = 0\} \quad (4.27)$$

$$S_1 = \{\tilde{h}_1 = \dots = \tilde{h}_{n-1} = 0, \sigma_0 = 1\}. \quad (4.28)$$

The global residue theorem thus reads

$$\sum_{p \in S_0} \text{Res}_{\{(\tilde{h}_1), \dots, (\tilde{h}_{n-1}), (\sigma_0^m)\}, p} \Gamma + \sum_{p' \in S_1} \text{Res}_{\{(\tilde{h}_1), \dots, (\tilde{h}_{n-1}), (\sigma_0 - 1)\}, p'} \Gamma = 0 \quad (4.29)$$

The second term actually equals to the sum of residues of all the finite poles of Ω , which can be seen by writing it in terms of contour integral and then integrate out σ_0 :

$$\begin{aligned} \sum_{p' \in S_1} \text{Res}_{\{(\tilde{h}_1), \dots, (\tilde{h}_{n-1}), (\sigma_0 - 1)\}, p'} \Gamma &= \oint_{\tilde{h}_1 = \dots = \tilde{h}_{n-1} = 0, \sigma_0 = 1} \frac{\mathcal{N}_\infty}{\tilde{h}_1 \dots \tilde{h}_{n-1} \cdot \hat{h}_0} d\sigma_1 \wedge \dots \wedge d\sigma_{n-1} \wedge d\sigma_0 \\ &= \oint_{h_1 = \dots = h_{n-1} = 0} \frac{1}{h_1 \dots h_{n-1}} \frac{\mathcal{N}_\infty}{\mathcal{D}} d\sigma_1 \wedge \dots \wedge d\sigma_{n-1} \\ &= \sum_{\text{finite poles}} \text{Res}_{\{h_1, \dots, h_{n-1}\}, p} \Omega. \end{aligned} \quad (4.30)$$

The first term, on the other hand, is of the form meeting conditions of Conjecture 2.1 and therefore can be computed accordingly. Thus the problem of finding the sum of residues of finite poles of Ω has been turned into that for Γ which can be done using Conjecture 2.1.

4.4 Residues at spurious poles

Now we are left with the last residue at the pole $\sigma_1 = \dots = \sigma_{n-1} = 1$. We call this pole “spurious” since it is not present in the solution to the original scattering equations. Notice that here we are interested in the residue *at a particular pole*, and thus the previous ansatz (2.3) does not apply because it computes the sum of residues of *all* finite poles. To deal with the current case, first we need to consider another homogenized version of the polynomial scattering equations $\{\hat{h}_i\}$. From $\{\hat{h}_i\}$ we then construct a differential operator $\hat{\mathbb{D}}$.⁵ This operator is conjectured to give the residue at the spurious pole.

⁵This may sound the same with the process dealing with the finite poles, i.e $\{h_i\} \rightarrow \{\tilde{h}_i\} \rightarrow \mathbb{D}$, but actually both $\{\hat{h}_i\}$ and $\hat{\mathbb{D}}$ are constructed in ways different from the previous versions, as explained in detail in the following.

As we have mentioned above, the residue at a given pole depends only on the local information of that pole. Hence a natural step to take is to parallel transport the coordinate system by $\sigma_i \rightarrow \sigma_i + 1$ such that the pole becomes the origin. This is convenient for later discussion in this part.

Here we construct $\{\hat{h}_i\}$ from $\{h_i\}$. In the neighbourhood of the pole, now at the origin, each polynomial h_i can be separated into its leading order term $L(h_i)$ and higher order terms $H(h_i)$. Let d_{L_i} be the degree of the leading order term $L(h_i)$. For each term m in $H(h_i)$ of degree d_m , we construct \hat{m} by substituting some of the σ_i factors with constants (w.r.t σ 's) t_i 's such that \hat{m} has a power in σ 's the same with $L(h_i)$.⁶ In this way all higher order terms are degraded to the degree of the leading term, with which we then define the homogenous polynomials $\{\hat{h}_i\}$

$$H(h_i) = \sum_a m_a(\sigma_i) \rightarrow \hat{H}(h_i) = \sum_a \hat{m}_a(\sigma_i, t_i), \quad \hat{h}_i = L(h_i) + \hat{H}(h_i). \quad (4.31)$$

Now we proceed to define the operator $\hat{\mathbb{D}}$. Those hatted polynomials \hat{h}_i 's are homogeneous, so exactly as in Section (2) we could define the differential operator \mathbb{D} corresponding to the integration with \hat{h}_i 's. \mathbb{D} is a differential operator with coefficients $a_{s_0 \dots s_{n-1}}$ being functions of t_i 's. Namely,

$$\oint_{\hat{h}_1 = \dots = \hat{h}_{n-1} = 0} \frac{d\sigma_1 \wedge \dots \wedge d\sigma_{n-1}}{\hat{h}_1 \dots \hat{h}_{n-1}} \iff \mathbb{D} = \sum_{\{s_i\}} a_{s_1 \dots s_{n-1}}(t_j) \partial_{r_1}^{s_1} \partial_{r_2}^{s_2} \dots \partial_{r_{n-1}}^{s_{n-1}}. \quad (4.32)$$

From \mathbb{D} we now define the differential operator $\hat{\mathbb{D}}$ that acts on an arbitrary function \mathcal{F} in the following way,

$$\hat{\mathbb{D}}\mathcal{F} = \sum_{\{s_0, \dots, s_{n-1}\}} \partial_{r_1}^{s_1} \partial_{r_2}^{s_2} \dots \partial_{r_{n-1}}^{s_{n-1}} (a_{s_1 \dots s_{n-1}}(\sigma_j) \mathcal{F}). \quad (4.33)$$

Based on large amount of numerical test, we propose the following conjecture to compute the residue at a particular pole:

Conjecture 4.2. *If the solution of $\langle L(h_0) = 0, L(h_1) = 0, \dots, L(h_{n-1}) = 0 \rangle$ is an isolated point p , then the residue at this pole defined by*

$$Res_p[\mathcal{R}(\sigma_i)] := \oint_{h_1 = \dots = h_{n-1} = 0} \frac{d\sigma_1 \wedge \dots \wedge d\sigma_{n-1}}{h_1 \dots h_{n-1}} \mathcal{R}(\sigma_i) \quad (4.34)$$

⁶For instance, if $m = \sigma_1 \sigma_2 \sigma_3^2$ and $d_{L_i} = 2$, the replacement can be $\hat{m} = t_1 t_2 \sigma_3^2$. Or replace $\sigma_2 \cdot \sigma_2$ with $t_1 \sigma_2$ if $d_{L_i} = 1$. Note that the particular choice of the replaced variables does not affect the final result. We have yet not found a direct proof for this property, but have checked it against many numerical or analytical examples.

can be obtained by $\hat{\mathbb{D}}$ in the following way

$$\text{Res}_p[\mathcal{R}(\sigma_i)] = \hat{\mathbb{D}}[\mathcal{R}(\sigma_i)] \quad (4.35)$$

where $\mathcal{R}(\sigma_i)$ is a holomorphic function in the neighborhood of the origin.

Besides a large number of numerical checks, this conjecture has also passed the analytical check for the one-loop super Yang-Mills integrand with four external particles.

⁷

5 Four-point one-loop SYM integrand

In this section, we further exemplify our algorithm by studying the super Yang-Mills one-loop amplitude for four particles. The four-point SYM amplitude is known to be particularly simple, since it only has one non-vanishing helicity configuration that is MHV. The planar 4-point amplitude at one loop is captured by the BDS ansatz [53] while the non-planar contribution is shown to be related to the planar part in [54–58].

⁷It is possible to get some intuition about this conjecture by considering a simpler scenario. Let f_i be a set of inhomogeneous polynomials and $L(f_i)$ their respective leading order terms. The polynomials \hat{f}_i are defined the same way as in (4.31). Suppose there exists a transformation matrix \mathbb{M} such that,

$$\begin{pmatrix} \vdots \\ L(f_i) \\ \vdots \end{pmatrix} = \mathbb{M}(\sigma_i) \begin{pmatrix} \vdots \\ f_i \\ \vdots \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ L(f_i) \\ \vdots \end{pmatrix} = \mathbb{M}(t_i) \begin{pmatrix} \vdots \\ \hat{f}_i \\ \vdots \end{pmatrix}.$$

Then we have, for a rational function \mathcal{F}

$$\begin{aligned} \oint_{\dots \cap (f_i) \cap \dots} \frac{d\sigma_1 \dots d\sigma_i \dots}{f_1 \dots f_i \dots} \mathcal{F} &= \oint_{\dots \cap (L(f_i)) \cap \dots} \frac{\det \mathbb{M}(\sigma_i) d\sigma_1 \dots d\sigma_i \dots}{L(f_1) \dots L(f_i) \dots} \mathcal{F}, \\ \oint_{\dots \cap (\hat{f}_i) \cap \dots} \frac{d\sigma_1 \dots d\sigma_i \dots}{\hat{f}_1 \dots \hat{f}_i \dots} \mathcal{F} &= \oint_{\dots \cap (L(f_i)) \cap \dots} \frac{\det \mathbb{M}(t_i) d\sigma_1 \dots d\sigma_i \dots}{L(f_1) \dots L(f_i) \dots} \mathcal{F}. \end{aligned}$$

Let \mathbb{D} , \mathbb{D}_L and $\hat{\mathbb{D}}$ denote the differential operators corresponding to the integrations with f_i , $L(f_i)$ and \hat{f}_i in the denominator respectively. It can be verified that,

$$\mathbb{D}\mathcal{F} = \mathbb{D}_L(\det \mathbb{M}(\sigma_i)\mathcal{F}) = \mathbb{D}_L\left(\lim_{t_i \rightarrow \sigma_i} \det \mathbb{M}(t_i)\mathcal{F}\right) = \hat{\mathbb{D}}\left(\lim_{t_i \rightarrow \sigma_i} \det \mathbb{M}(t_i)\mathcal{F}\right).$$

Of course it is not obvious whether such a transformation exists and we can not prove our conjecture in general.

Here we are interested in mainly the mathematical properties of the CHY-form of this one-loop integrand and its explicit expression is given in [46, 47],

$$\mathcal{I}_4^{l=1} = \oint_{f_1=\dots=f_3=0} \frac{d\sigma_1 \cdots d\sigma_3}{f_1 \cdots f_3} \text{Pf}(M_4) PT_4 \prod_{i=1}^3 \frac{1}{\sigma_i}. \quad (5.1)$$

The well-known Parke-Taylor factor reads

$$PT_4 = \sum_{\gamma} \frac{1}{\sigma_{\gamma(1)}(\sigma_{\gamma(1)} - \sigma_{\gamma(2)})(\sigma_{\gamma(2)} - \sigma_{\gamma(3)})(\sigma_{\gamma(3)} - \sigma_{\gamma(4)})}, \quad (5.2)$$

where we sum over all the S_4 permutations of the indices. The Pfaffian in this case is simply a constant. The polynomial scattering equations are easy to construct as in Section 4.1. Substituting the Vandermonde determinant and the Parke-Taylor factor, we arrive at the simple contour integral form of the 4-point integrand,

$$\mathcal{I}_4^{l=1} = \oint_{h_1=h_2=h_3=0} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1 h_2 h_3} \mathcal{R}_4 - \oint_{\sigma_1=\sigma_2=\sigma_3=1} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1 h_2 h_3} \mathcal{R}_4, \quad (5.3)$$

where the polynomials h_i 's are given in (4.7) and we have chosen the gauge $\sigma_4 = 1$. The function \mathcal{R}_4 reads,

$$\mathcal{R}_4 = \frac{(\sigma_2 - 1)(\sigma_1 - \sigma_3) [\sigma_1^2 \sigma_3 + \sigma_2 \sigma_3 + \sigma_1 \sigma_2 (\sigma_2 + \sigma_3 (\sigma_3 - 4))]}{\sigma_1 \sigma_2 \sigma_3}. \quad (5.4)$$

We consider the integral around the origin first. The integrand is a sum of meromorphic functions and the terms can be put into three categories, depending on their singularities: (1) functions that have only poles originating from the scattering equations, i.e poles of h_i 's; (2) functions that have poles originating from h_i 's and other factors, such as σ_i 's, but are regular at infinity; (3) functions that have poles at infinity. The residues of those in the first category are obviously zero and we drop these terms from now on. There are only four surviving terms and we denote them as $\mathcal{R}_4 = \sum_{i=1}^4 \mathcal{R}_4^{(i)}$. The first three terms read,

$$\mathcal{R}_4^{(1)} = \frac{(\sigma_2 - 1)\sigma_3}{\sigma_1}, \quad \mathcal{R}_4^{(2)} = \frac{(\sigma_1 - \sigma_3)\sigma_1}{\sigma_2}, \quad \mathcal{R}_4^{(3)} = \frac{\sigma_1 \sigma_2 (\sigma_2 - 1)}{\sigma_3}. \quad (5.5)$$

These terms do not have non-zero residues at infinity. The residue corresponding to the last one, however, is non-vanishing at infinity and needs to be taken care of differently,

$$\mathcal{R}_4^{(4)} = \sigma_2 \sigma_3 (\sigma_3 - \sigma_1). \quad (5.6)$$

The meromorphic forms corresponding to $\mathcal{R}_4^{(i)}$ are

$$\Gamma_1 = \frac{(\sigma_2 - 1)\sigma_3}{h_1 h_2 h_3 \cdot \sigma_1} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3, \quad (5.7)$$

$$\Gamma_2 = \frac{(\sigma_1 - \sigma_3)\sigma_1}{h_1 h_2 h_3 \cdot \sigma_2} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3 \quad (5.8)$$

$$\Gamma_3 = \frac{\sigma_1 \sigma_2 (\sigma_2 - 1)}{h_1 h_2 h_3 \cdot \sigma_3} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3, \quad (5.9)$$

$$\Gamma_4 = \frac{\sigma_2 \sigma_3 (\sigma_3 - \sigma_1)}{h_1 h_2 h_3} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3, \quad (5.10)$$

5.1 Computing the residues at finite poles

The residues associated with Γ_1 , Γ_2 , and Γ_3 can all be obtained the same way, following our conjecture, and here we just take Γ_1 as an example. First, we homogenize the factors h_1, h_2, h_3 with an auxiliary variable σ_0 and the residue associated with Γ_1 becomes,

$$\oint_{h_1=h_2=h_3=0} \Gamma_1 = \oint_{\tilde{h}_1=\tilde{h}_2=\tilde{h}_3=\sigma_0-1} \tilde{\Gamma}_1 \quad (5.11)$$

where

$$\tilde{\Gamma}_1 = \frac{(\sigma_2 - 1)\sigma_3}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \cdot \sigma_1 (\sigma_0 - 1)} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3 \wedge d\sigma_0. \quad (5.12)$$

The global residue theorem yields,

$$\oint_{\tilde{h}_1=\tilde{h}_2=\tilde{h}_3=\sigma_0-1} \tilde{\Gamma}_1 = -\text{Res}_{\{D_1, D_2, D_3, (\sigma_0)\}, p} \tilde{\Gamma}_1. \quad (5.13)$$

where $D_i = (\tilde{h}_i)$ ($i = 1, 2, 3$). All the intersecting divisors on the right-hand-side are generated by homogeneous polynomials whose common zero is assumed to be a single point, which thus must be the origin. The right-hand-side is corresponding to a third-order differential operator, as conjectured in 2.1,

$$\mathbb{D} = \sum_{\substack{0 \leq s_i \leq 3, \\ s_0 + s_1 + s_2 + s_3 = 3}} a_{s_0 s_1 s_2 s_3} \frac{\partial^{s_0}}{\partial \sigma_0^{s_0}} \frac{\partial^{s_1}}{\partial \sigma_1^{s_1}} \frac{\partial^{s_2}}{\partial \sigma_2^{s_2}} \frac{\partial^{s_3}}{\partial \sigma_3^{s_3}}. \quad (5.14)$$

There are 20 coefficients $a_{s_0 s_1 s_2 s_3}$ to be determined. The local duality theorem yields,

$$\mathbb{D}(\sigma_i \sigma_j \tilde{h}_0) = \mathbb{D}(\sigma_i \sigma_j \tilde{h}_1) = \mathbb{D}(\sigma_i \tilde{h}_2) = \mathbb{D} \tilde{h}_3 = 0 \quad 0 \leq i, j \leq 3. \quad (5.15)$$

These constraints have a huge redundancy and if one carefully computes the rank of the constraint matrix, it is in fact 19. The intersection number of the divisors in this case is 6 and this demands,

$$\mathbb{D} \left(\det \left[\frac{\partial \tilde{h}_i}{\partial \sigma_j} \right] \right) = 6. \quad (5.16)$$

Now we have 20 conditions that fix the coefficients completely. These constraints are simple and linear conditions and solving them possesses no difficulty at all. Substituting the solution into \mathbb{D} the residue associated with Γ_1 is given by

$$\begin{aligned} -\text{Res}_{\{D_1, D_2, D_3, (\sigma_0)\}, p} \tilde{\Gamma}_1 &= -\mathbb{D} \left[\frac{(\sigma_2 - 1)\sigma_3}{\sigma_0 - 1} \right] = -\mathbb{D} \left[\frac{-\sigma_3}{\sigma_0 - 1} \right] \\ &= \frac{1}{\ell \cdot k_1 \ell \cdot k_4 (k_{12} + \ell \cdot k_1 + \ell \cdot k_2)}. \end{aligned} \quad (5.17)$$

Likewise, the terms $\mathcal{R}_4^{(2)}$ and $\mathcal{R}_4^{(3)}$ give rise to the following residues respectively,

$$-\mathbb{D} \left[\frac{\sigma_1(\sigma_1 - \sigma_3)}{\sigma_0 - 1} \right] = -\mathbb{D} \left[\frac{\sigma_1^2}{\sigma_0 - 1} \right] = \frac{1}{\ell \cdot k_1 \ell \cdot k_2 (k_{23} + \ell \cdot k_2 + \ell \cdot k_3)}, \quad (5.18)$$

$$-\mathbb{D} \left[\frac{\sigma_1 \sigma_2 (\sigma_2 - 1)}{\sigma_0 - 1} \right] = -\mathbb{D} \left[\frac{\sigma_1 \sigma_2^2}{\sigma_0 - 1} \right] = \frac{1}{\ell \cdot k_2 \ell \cdot k_3 (k_{34} + \ell \cdot k_3 + \ell \cdot k_4)}. \quad (5.19)$$

5.2 Computing the residues at infinity

Now we discuss the term $\mathcal{R}_4^{(4)}$ which has a non-zero residue at infinity. The standard method to obtain the residue at infinity is discussed in Appendix B. Here we obtain the residue also by our ansatz. For that, we consider the form

$$\tilde{\Gamma}_4 = \frac{(\sigma_3 - \sigma_1) \sigma_2 \sigma_3}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \cdot \sigma_0 (\sigma_0 - 1)} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3 \wedge d\sigma_0. \quad (5.20)$$

According to the global residue theorem, we know that its residue at infinity is zero, i.e residues of all finite poles sum up to zero ([50, 51]). We choose the four divisors (for this 4-dim space) by

$$D_1 = (\tilde{h}_1), \quad D_2 = (\tilde{h}_2), \quad D_3 = (\tilde{h}_3), \quad D_4 = (\sigma_0(\sigma_0 - 1)). \quad (5.21)$$

The global residue theorem leads to

$$\begin{aligned} 0 &= \sum_p \text{Res}_{\{D_1, D_2, D_3, (\sigma_0(\sigma_0 - 1))\}, p} \tilde{\Gamma}_4 \\ &= \sum_p \text{Res}_{\{D_1, D_2, D_3, (\sigma_0)\}, p} \tilde{\Gamma}_4 + \sum_{p'} \text{Res}_{\{D_1, D_2, D_3, (\sigma_0 - 1)\}, p'} \tilde{\Gamma}_4. \end{aligned} \quad (5.22)$$

The second term, if integrated over σ_0 first, simply returns to the original integral

$$\oint_{h_1=h_2=h_3=0} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1 h_2 h_3} \mathcal{R}_4^{(4)}$$

Now according the ansatz 2.3, the first term is explicitly

$$\mathbb{D}_3 \left[\frac{\sigma_2 \sigma_3 (\sigma_3 - \sigma_1)}{\sigma_0 - 1} \right] = \mathbb{D}_3 \left[\frac{\sigma_2 \sigma_3^2}{\sigma_0 - 1} \right] = - \frac{1}{l \cdot k_3 (-k_{23} + l \cdot k_2 + l \cdot k_3) (-l \cdot k_4)}. \quad (5.23)$$

Thus from (5.22) we immediately get

$$\oint_{h_1=h_2=h_3=0} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1 h_2 h_3} \mathcal{R}_4^{(4)} = \frac{1}{l \cdot k_3 (-k_{23} + l \cdot k_2 + l \cdot k_3) (-l \cdot k_4)}$$

5.3 Computing the residues at spurious poles

We are now left with the spurious pole at $\sigma_1 = \dots = \sigma_{n-1} = 1$. The parameter transformation $\sigma_i \rightarrow \sigma_i + 1$ shifts the pole to the origin. The polynomial scattering equations are directly read off from (4.8). Unlike the first integral in (5.3), now we have to take all the terms in \mathcal{R}_4 into consideration. That is, we are computing the residue at the origin associate with the form,

$$\Gamma_{spurious} = \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1(\sigma_i + 1) h_2(\sigma_i + 1) h_3(\sigma_i + 1)} \mathcal{R}_4(\sigma_i + 1). \quad (5.24)$$

Following the discussion in Section 4.4, we homogenize the shifted polynomials $h(\sigma_i + 1)$ as follows,

$$\begin{aligned} \hat{h}_1 &= l_1 \sigma_1 + l_2 \sigma_2 + l_3 \sigma_3, \\ \hat{h}_2 &= -l_{12} \sigma_1 \sigma_2 - l_{13} \sigma_1 \sigma_3 - l_{23} \sigma_2 \sigma_3, \\ \hat{h}_3 &= -l_4 t_1 \sigma_2 \sigma_3 + k_{12} \sigma_1 \sigma_2 + k_{13} \sigma_1 \sigma_3 + k_{23} \sigma_2 \sigma_3. \end{aligned} \quad (5.25)$$

At the moment the quantity t_1 in \hat{h}_3 is regarded as a number and the degrees of the polynomials are $\deg(\hat{h}_1) = 1$, $\deg(\hat{h}_2) = 2$ and $\deg(\hat{h}_3) = 2$. The intersection number of the divisors $\hat{D}_i = (\hat{h}_i)$, ($i = 1, 2, 3$) is 4. Now we consider the following residue,

$$\text{Res}_{\{\hat{D}_1, \hat{D}_2, \hat{D}_3\}, p} \hat{\Gamma}_{spurious}, \quad \text{with} \quad \hat{\Gamma}_{spurious} = \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{\hat{h}_1 \hat{h}_2 \hat{h}_3} \mathcal{R}_4(\sigma_i + 1). \quad (5.26)$$

To obtain this residue using Conjecture 4.2, a second-order differential operator is to be worked out,

$$\mathbb{D} = a_{002} \frac{\partial^2}{\partial \sigma_3^2} + a_{011} \frac{\partial^2}{\partial \sigma_2 \partial \sigma_3} + a_{020} \frac{\partial^2}{\partial \sigma_2^2} + a_{101} \frac{\partial^2}{\partial \sigma_1 \partial \sigma_3} + a_{110} \frac{\partial^2}{\partial \sigma_1 \partial \sigma_2} + a_{200} \frac{\partial^2}{\partial \sigma_1^2} \quad (5.27)$$

The local residue theorem and the intersection number conditions demand,

$$\mathbb{D}(\sigma_1 \hat{h}_1) = 0, \quad \mathbb{D}(\sigma_2 \hat{h}_1) = 0, \quad \mathbb{D}(\sigma_3 \hat{h}_1) = 0, \quad \mathbb{D} \hat{h}_2 = 0, \quad \mathbb{D} \hat{h}_3 = 0, \quad \mathbb{D} \hat{\mathcal{J}} = 4. \quad (5.28)$$

where $\hat{\mathcal{J}} \equiv \det(\partial_i \hat{h}_j)$. These constraints are easily solved. Note that the condition matrix here is invertible as $t_1 \rightarrow 0$. Substituting $t_1 = \sigma_1$ back in the expressions for the coefficients a_{ijk} 's, the ansatz (4.35) for the inhomogenous case leads to,

$$\begin{aligned} & \hat{\mathbb{D}}(\mathcal{I}_4(\sigma_i + 1)) \\ & \equiv \frac{\partial^2}{\partial \sigma_3^2} [a_{002} \mathcal{R}_4(\sigma_i + 1)] + \frac{\partial^2}{\partial \sigma_2 \partial \sigma_3} [a_{011} \mathcal{R}_4(\sigma_i + 1)] + \frac{\partial^2}{\partial \sigma_2^2} [a_{020} \mathcal{R}_4(\sigma_i + 1)] \\ & \quad + \frac{\partial^2}{\partial \sigma_1 \partial \sigma_3} [a_{101} \mathcal{R}_4(\sigma_i + 1)] + \frac{\partial^2}{\partial \sigma_1 \partial \sigma_2} [a_{110} \mathcal{R}_4(\sigma_i + 1)] + \frac{\partial^2}{\partial \sigma_1^2} [a_{200} \mathcal{R}_4(\sigma_i + 1)] \\ & = \text{Res}_{\{(h_1(\sigma_i+1)), (h_2(\sigma_i+1)), (h_3(\sigma_i+1))\}, p} \Gamma_{\text{spurious}} \end{aligned} \quad (5.29)$$

Fortunately, the explicit expressions for the coefficients a_{ijk} are not necessary for computing this residue. For instance, consider the term

$$\frac{\partial^2}{\partial \sigma_1 \partial \sigma_3} a_{101} \mathcal{R}_4(\sigma_i + 1) \Big|_{\sigma_i=0} = \left[\frac{1}{\sigma_2 + 1} + \frac{1}{(\sigma_3 + 1)^2} + \frac{1}{\sigma_1 + 1} - 3 \right] \frac{\partial a_{101}(\sigma_1)}{\partial \sigma_1} \Big|_{\sigma_i=0} \quad (5.30)$$

This vanishes since $\frac{\partial a_{101}(\sigma_1)}{\partial \sigma_1}$ is holomorphic in the neighbourhood of the origin and the factor multiplying this derivative is zero at the origin. This is true for all the terms in the action of $\hat{\mathbb{D}}$ when the condition matrix is invertible. Hence the residue at the spurious pole is vanishing.

5.4 Summary of the four-point integrand

Now we conclude this section by summarizing the CHY-form for the 4-point 1-loop SYM integrand evaluated by our ansatz. Explicitly, the final result reads,

$$\begin{aligned} \mathcal{I}_4^{l=1} &= \oint_{h_1=h_2=h_3=0} (\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4) \\ &= \frac{1}{\ell \cdot k_1 \ell \cdot k_4 (\ell \cdot k_1 + \ell \cdot k_2 + k_{12})} + \frac{1}{\ell \cdot k_1 \ell \cdot k_2 (\ell \cdot k_2 + \ell \cdot k_3 + k_{23})} \\ & \quad - \frac{1}{\ell \cdot k_2 \ell \cdot k_3 (\ell \cdot k_1 + \ell \cdot k_2 - k_{12})} - \frac{1}{\ell \cdot k_3 \ell \cdot k_4 (\ell \cdot k_2 + \ell \cdot k_3 - k_{23})}. \end{aligned} \quad (5.31)$$

The four terms in this expression have a one-to-one correspondence with the four forward-limit channels in the Q-cut representation of the same amplitude [59, 60]. This is a consequence of the string origin of the CHY representations. The singular behaviour of the CHY-form is inherited from the worldsheet factorization structure [43, 44, 48], which is naturally related to the forward limit.

6 Outlooks

So far we have developed a differential operator for the residue with respect to a general meromorphic form and exploited it in the study of the four-point CHY expressions at one loop. An immediate follow-up direction is to probe the one-loop CHY-form for higher points. Starting from 5-points, the integrands of amplitudes grow more complicated, in particular, nontrivial Pfaffians enter in the integrand in SYM and SUGRA. Nevertheless, these factors are rational functions and our method is expected to apply to higher-point integrands comfortably. Having collected more analytical data for higher-point expressions, other ways may be discovered to determine the exact form of the differential operator, without solving the corresponding constraints by brute force.

Our method also serves as a useful tool to investigate the higher-loop CHY-forms in Yang-Mills and gravity theory, as well as to explore the non-planar regime of these theories where new symmetries are likely to be hiding. At two loops, the construction of the integral basis, involving the integration-by-parts (IBP) relations among loop integrands, remains an interesting open question. CHY-forms may be a new playground for such construction and the conjectures for residues hopefully help us in finding similar relations at the level of CHY expressions. Many aspects of Yang-Mills and gravity outside the large-N limit are still uncharted at the moment. Constructing CHY-like representations for non-planar amplitudes is certainly of importance while the string origin of such representations may make some symmetries and algebraic structures, which are otherwise hard to observe, manifest.

Furthermore, this method also finds natural applications in a variety of aspects of scattering amplitudes, such as the Grassmannian integral form and the generalized unitarity cut.

A Numerical Checks for the Ansatz

In this appendix we present numerical checks of the ansatz for the differential operator used in the computation of the residues. To check the ansatz, we consider different ideals of polynomials $\langle h_1, h_2, \dots, h_n \rangle$ and obtain the positions of the intersection point by solving the corresponding algebraic equations numerically. For some given integrand $g(\sigma_i)$ we compute its residues at these intersection points, both directly and using our conjectures. In all the examples we have considered, the numerical results obtained from both methods match beautifully.

To compute the residue directly at a given intersection point, we have to consider the nature of this intersection point first. There are two types of isolated intersection points, the singular ones and the non-singular ones. For a non-singular isolated point $p \in (h_1) \cap (h_2) \cdots \cap (h_n)$, the residue is

$$\text{Res}_p \frac{g(\sigma_i) d\sigma_1 \wedge \cdots \wedge d\sigma_n}{h_1 \cdots h_n} = \frac{g}{\mathcal{J}} \Big|_p, \quad (\text{A.1})$$

where $\mathcal{J}|_p = \det \left(\frac{\partial f_i}{\partial \sigma_j} \right) \Big|_p$ is nonzero for the non-singular point. For a singular isolated intersection point, we need to perform a deformation first. To guarantee that no information of the singularity gets lost, we need a semiuniversal deformation and a deformation generated by the Tjurina algebra [61, 62] is a nice candidate. For a singular isolated point defined by a set of generators of the ideal $\langle h_1, h_2, \dots, h_n \in \mathcal{O}_{\mathbb{C}^n, p} \rangle$, the Tjurina algebra is

$$Tj = \mathcal{O}_{\mathbb{C}^n, p} / \langle h_1 \vec{e}_1, \dots, h_1 \vec{e}_n, \dots, h_n \vec{e}_1, \dots, h_n \vec{e}_n, \partial_{\sigma_1} \vec{h}, \dots, \partial_{\sigma_n} \vec{h} \rangle,$$

where \vec{h} is the n -column (h_1, h_2, \dots, h_n) and \vec{e}_i is the unit n -column with its i -th element set to be 1. In this case, the algebra has a finite number of generators $\vec{g}_1, \dots, \vec{g}_\tau$, where τ denotes the total number of the generators known as the Tjurina number. The semiuniversal deformation reads,

$$\vec{F}(\sigma, t) = \vec{h}(\sigma) + \sum_{i=1}^{\tau} t_i \vec{g}_i.$$

In principle one is supposed to perform such a deformation. However, in practice, we only have to use a sub-space in the Tjurina algebra such that the singular point of degree d is decoupled into d separated intersection points p^i . This process is easy to implement numerically by simply choosing each parameter t_i to be a very small number. After

deforming the singular point, we sum up the residue over all the separated intersection points and obtain the residue at the original point,

$$\text{Res}_p \frac{g(\sigma_i) d\sigma_1 \cdots d\sigma_n}{h_1 \cdots h_n} = \sum_{i=1}^d \frac{g}{\mathcal{J}} \Big|_{p^i} . \quad (\text{A.2})$$

In numerical computations, the high precision of the results is guaranteed as long as the deformation parameters t_i are sufficiently small.

A.1 Homogeneous ideals

In this section we take 6 homogeneous ideals as our examples to test Conjecture 2.1. (We have tested our ansatz against a lot more examples and believe our method to be quite robust.) The coefficients in these ideals are randomly generated integers. Half of the ideals contain 3 variables and the degrees of the polynomials in them range from 4 to 6. The rest examples are ideals consisting of polynomials of degree 2 and the number of variables ranges from 4 to 6. The integrand is chosen to be $g(\sigma_i) = \frac{\sigma_i+1}{\sum_{i=1}^n \sigma_i+1}$ for all examples, where n denotes the number of the variables in the ideal.⁸ The ideals are given below and the residues computed using the two methods are listed in Table 1.

$$\begin{aligned} \mathcal{I}_3^4 = & \langle 7\sigma_1^4 + 7\sigma_2\sigma_1^3 + 9\sigma_3\sigma_1^3 + 2\sigma_2^2\sigma_1^2 + 18\sigma_3^2\sigma_1^2 + 11\sigma_2\sigma_3\sigma_1^2 + 17\sigma_2^3\sigma_1 + 18\sigma_3^3\sigma_1 \\ & + 23\sigma_2\sigma_3^2\sigma_1 + 14\sigma_2^2\sigma_3\sigma_1 + 9\sigma_2^4 + 16\sigma_3^4 + 20\sigma_2\sigma_3^3 + 19\sigma_2^2\sigma_3^2 + 12\sigma_2^3\sigma_3, \\ & \sigma_1^4 + 23\sigma_2\sigma_1^3 + 12\sigma_3\sigma_1^3 + 13\sigma_2^2\sigma_1^2 + 22\sigma_3^2\sigma_1^2 + 22\sigma_2\sigma_3\sigma_1^2 + 6\sigma_2^3\sigma_1 + 16\sigma_3^3\sigma_1 \\ & + 20\sigma_2\sigma_3^2\sigma_1 + 16\sigma_2^2\sigma_3\sigma_1 + 18\sigma_2^4 + 19\sigma_3^4 + 3\sigma_2\sigma_3^3 + 11\sigma_2^2\sigma_3^2 + 9\sigma_2^3\sigma_3, \\ & \sigma_1^4 + 19\sigma_2\sigma_1^3 + 20\sigma_3\sigma_1^3 + 12\sigma_2^2\sigma_1^2 + 19\sigma_3^2\sigma_1^2 + 22\sigma_2\sigma_3\sigma_1^2 + 12\sigma_2^3\sigma_1 + 20\sigma_3^3\sigma_1 \\ & + 10\sigma_2\sigma_3^2\sigma_1 + 17\sigma_2^2\sigma_3\sigma_1 + 5\sigma_2^4 + 3\sigma_3^4 + 11\sigma_2\sigma_3^3 + 17\sigma_2^2\sigma_3^2 + 22\sigma_2^3\sigma_3 \rangle, \end{aligned}$$

⁸This function g is chosen such that none of the derivatives ∂_i ever acts trivially on the integrand.

$$\begin{aligned}
\mathcal{I}_3^5 = & \langle 6\sigma_1^5 + 2\sigma_2\sigma_1^4 + 3\sigma_3\sigma_1^4 + 9\sigma_2^2\sigma_1^3 + 6\sigma_3^2\sigma_1^3 + 4\sigma_2\sigma_3\sigma_1^3 + 3\sigma_2^3\sigma_1^2 \\
& + 12\sigma_3^3\sigma_1^2 + 12\sigma_2\sigma_3^2\sigma_1^2 + 2\sigma_2^2\sigma_3\sigma_1^2 + 13\sigma_2^4\sigma_1 + 8\sigma_3^4\sigma_1 + 11\sigma_2\sigma_3^3\sigma_1 \\
& + 9\sigma_2^2\sigma_3^2\sigma_1 + 4\sigma_2^3\sigma_3\sigma_1 + 8\sigma_2^5 + 7\sigma_3^5 + 5\sigma_2\sigma_3^4 + 6\sigma_2^2\sigma_3^3 + 5\sigma_2^3\sigma_3^2, \\
& 7\sigma_1^5 + 10\sigma_2\sigma_1^4 + 3\sigma_3\sigma_1^4 + \sigma_2^2\sigma_1^3 + 10\sigma_3^2\sigma_1^3 + 6\sigma_2\sigma_3\sigma_1^3 + 13\sigma_2^3\sigma_1^2 \\
& + 13\sigma_3^3\sigma_1^2 + 9\sigma_2\sigma_3^2\sigma_1^2 + 4\sigma_2^2\sigma_3\sigma_1^2 + 5\sigma_2^5 + \sigma_2^4\sigma_1 + 13\sigma_3^4\sigma_1 + 3\sigma_2\sigma_3^3\sigma_1 \\
& + 2\sigma_2^2\sigma_3^2\sigma_1 + 9\sigma_2^3\sigma_3\sigma_1 + 10\sigma_3^5 + 10\sigma_2\sigma_3^4 + 13\sigma_2^2\sigma_3^3 + 11\sigma_2^3\sigma_3^2 + 8\sigma_2^4\sigma_3, \\
& 2\sigma_1^5 + 10\sigma_2\sigma_1^4 + 4\sigma_3\sigma_1^4 + \sigma_2^2\sigma_1^3 + 2\sigma_3^2\sigma_1^3 + 4\sigma_2\sigma_3\sigma_1^3 + 8\sigma_2^3\sigma_1^2 \\
& + 8\sigma_3^3\sigma_1^2 + 3\sigma_2\sigma_3^2\sigma_1^2 + 5\sigma_2^4\sigma_1 + 8\sigma_3^4\sigma_1 + 7\sigma_2\sigma_3^3\sigma_1 + 9\sigma_2^2\sigma_3^2\sigma_1 \\
& + 13\sigma_2^3\sigma_3\sigma_1 + 8\sigma_2^5 + 12\sigma_3^5 + 7\sigma_2\sigma_3^4 + 2\sigma_2^3\sigma_3^2 + 7\sigma_2^4\sigma_3 \rangle,
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_3^6 = & \langle 3\sigma_1^6 + 4\sigma_2\sigma_1^5 + 2\sigma_3\sigma_1^5 + 5\sigma_2^2\sigma_1^4 + 2\sigma_3^2\sigma_1^4 + 4\sigma_2\sigma_3\sigma_1^4 + \sigma_2^3\sigma_1^3 + \sigma_3^3\sigma_1^3 + 2\sigma_2\sigma_3^2\sigma_1^3 \\
& + \sigma_2^2\sigma_3\sigma_1^3 + 2\sigma_2^4\sigma_1^2 + 3\sigma_3^4\sigma_1^2 + 5\sigma_2\sigma_3^3\sigma_1^2 + \sigma_2^2\sigma_3^2\sigma_1^2 + \sigma_2^3\sigma_3\sigma_1^2 + 3\sigma_2^5\sigma_1 + 4\sigma_2\sigma_3^4\sigma_1 \\
& + 3\sigma_2^2\sigma_3^3\sigma_1 + \sigma_2^3\sigma_3^2\sigma_1 + 4\sigma_2^4\sigma_3\sigma_1 + 3\sigma_2^6 + 2\sigma_3^6 + 3\sigma_2\sigma_3^5 + 4\sigma_2^2\sigma_3^4 + 3\sigma_2^3\sigma_3^3 + 3\sigma_2^4\sigma_3^2, \\
& 4\sigma_1^6 + 5\sigma_2^6 + 3\sigma_3^6 + 2\sigma_2^2\sigma_1^4 + 4\sigma_3^2\sigma_1^4 + \sigma_2\sigma_3\sigma_1^4 + 3\sigma_2^3\sigma_1^3 + 3\sigma_2\sigma_3^2\sigma_1^3 + 4\sigma_2^4\sigma_1^2 + 2\sigma_2\sigma_3^3\sigma_1^2 \\
& + 2\sigma_3^4\sigma_1^2 + 2\sigma_2^2\sigma_3^2\sigma_1^2 + 3\sigma_2^3\sigma_3\sigma_1^2 + 3\sigma_2^5\sigma_1 + \sigma_2\sigma_3^4\sigma_1 + 2\sigma_2^4\sigma_3\sigma_1 + 2\sigma_2^2\sigma_3^4 + 4\sigma_2^3\sigma_3^3 + 3\sigma_2^5\sigma_3, \\
& 5\sigma_1^6 + 2\sigma_2\sigma_1^5 + 5\sigma_3\sigma_1^5 + \sigma_2^2\sigma_1^4 + 5\sigma_3^2\sigma_1^4 + 4\sigma_2^3\sigma_1^3 + \sigma_3^3\sigma_1^3 + 4\sigma_2\sigma_3^2\sigma_1^3 + 5\sigma_2^2\sigma_3\sigma_1^3 + 3\sigma_2\sigma_3^3\sigma_1^2 \\
& + 4\sigma_3^4\sigma_1^2 + 2\sigma_2^2\sigma_3^2\sigma_1^2 + 3\sigma_2^3\sigma_3\sigma_1^2 + 2\sigma_2^5\sigma_1 + 5\sigma_3^5\sigma_1 + 4\sigma_2^2\sigma_3^3\sigma_1 + 2\sigma_2^3\sigma_3^2\sigma_1 + 5\sigma_2^4\sigma_3\sigma_1 + 4\sigma_2^6 \\
& + 2\sigma_3^6 + 2\sigma_2\sigma_3^5 + \sigma_2^3\sigma_3^3 + 5\sigma_2^4\sigma_3^2 + 4\sigma_2^5\sigma_3 \rangle.
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_4^2 = & \langle 4\sigma_1^2 + 4\sigma_2\sigma_1 + 4\sigma_3\sigma_1 + 3\sigma_4\sigma_1 + \sigma_2^2 + 5\sigma_3^2 + \sigma_4^2 + 2\sigma_2\sigma_3 + 2\sigma_2\sigma_4 + \sigma_3\sigma_4, \\
& \sigma_1^2 + \sigma_2\sigma_1 + 2\sigma_3\sigma_1 + 4\sigma_4\sigma_1 + 5\sigma_2^2 + \sigma_4^2 + 4\sigma_2\sigma_3 + 5\sigma_2\sigma_4 + 2\sigma_3\sigma_4, \\
& 4\sigma_1^2 + 2\sigma_2\sigma_1 + 2\sigma_3\sigma_1 + 2\sigma_4\sigma_1 + 3\sigma_2^2 + 4\sigma_3^2 + 5\sigma_4^2 + 4\sigma_2\sigma_3 + 2\sigma_2\sigma_4 + 3\sigma_3\sigma_4, \\
& 2\sigma_1^2 + \sigma_2\sigma_1 + 3\sigma_3\sigma_1 + 5\sigma_2^2 + 5\sigma_3^2 + 5\sigma_2\sigma_3 + 2\sigma_2\sigma_4 + 5\sigma_3\sigma_4 \rangle,
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_5^2 = & \langle 5\sigma_1^2 + 5\sigma_2\sigma_1 + 3\sigma_3\sigma_1 + \sigma_4\sigma_1 + 2\sigma_5\sigma_1 + \sigma_2^2 + 5\sigma_5^2 + 5\sigma_2\sigma_4 + \sigma_3\sigma_4 + 5\sigma_4\sigma_5, \\
& 2\sigma_1^2 + 3\sigma_2\sigma_1 + \sigma_3\sigma_1 + 4\sigma_4\sigma_1 + 5\sigma_5\sigma_1 + \sigma_2^2 + 3\sigma_3^2 + 5\sigma_4^2 \\
& + \sigma_5^2 + 3\sigma_2\sigma_3 + 5\sigma_2\sigma_4 + 5\sigma_3\sigma_4 + 2\sigma_3\sigma_5 + 3\sigma_4\sigma_5, \\
& 5\sigma_1^2 + \sigma_3\sigma_1 + 3\sigma_4\sigma_1 + 4\sigma_3^2 + 2\sigma_4^2 + 2\sigma_5^2 + 2\sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_2\sigma_5 + \sigma_3\sigma_5 + 2\sigma_4\sigma_5, \\
& \sigma_1^2 + \sigma_2\sigma_1 + \sigma_3\sigma_1 + 2\sigma_4\sigma_1 + 3\sigma_5\sigma_1 + 5\sigma_3^2 + 5\sigma_4^2 + 2\sigma_2\sigma_3 \\
& + 3\sigma_2\sigma_4 + 4\sigma_3\sigma_4 + 4\sigma_2\sigma_5 + 5\sigma_3\sigma_5 + 4\sigma_4\sigma_5, \\
& \sigma_2^2 + 5\sigma_1\sigma_2 + 4\sigma_3\sigma_2 + 3\sigma_4\sigma_2 + 2\sigma_5\sigma_2 + 3\sigma_3^2 + 2\sigma_4^2 + 4\sigma_1\sigma_3 + 2\sigma_1\sigma_4 + 3\sigma_3\sigma_4 + \sigma_3\sigma_5 + 5\sigma_4\sigma_5 \rangle,
\end{aligned}$$

Table 1: The residues evaluated numerically at the intersection points are shown in the second column while the ones calculated by the conjecture are shown in the last column. Since the conjecture bypasses solving the equations numerically, the results in the last column are semi-analytical. Although not obvious at all, the values in the second and the last columns match up to computational precision.

Ideal	Numerical	Residue from the ansatz
\mathcal{I}_3^4	-0.00196517	$-\frac{338369835974858276439763339608916939488780049506461387859}{172234849446370037753709508680189823992399891128510899390423}$
\mathcal{I}_3^5	1.33251	$\frac{185305421348253037675212877549168515918001249681372449355363515406576810500662817}{139071748905554975365797175643454675160373469042145930039888932779199639222995099}$
\mathcal{I}_3^6	593.819	$\frac{219551617079855111701999235055855939627650975665549069197122883934238537987536968925296142}{370553184196614950880443018799517897307360951546021926710875320859728913170901077938477}$
\mathcal{I}_4^2	-0.0023375	$-\frac{2153841252590208111}{921484430712569847077}$
\mathcal{I}_5^2	-0.0112606	$-\frac{473520568018181185074562119595694532674773643}{42044737913776287413645978868169636186479087179}$
\mathcal{I}_6^2	-0.00895405	$-\frac{519880741176657236666450793152822696600712134946666110128882345754}{65989452607833385784278736240681558611734904658085876275390766034}$ $-\frac{5967246404862841666772950558667081567881828770924504194620658839691}{2630580897251182112405482125334207621804213585965755701639782720971}$

$$\begin{aligned}
\mathcal{I}_6^2 = & \langle \sigma_1^2 + 3\sigma_2\sigma_1 + 3\sigma_3\sigma_1 + 4\sigma_5\sigma_1 + 5\sigma_6\sigma_1 + 5\sigma_2^2 + \sigma_3^2 + \sigma_5^2 + 4\sigma_6^2 + 3\sigma_2\sigma_3 \\
& + 3\sigma_2\sigma_4 + 5\sigma_3\sigma_4 + 4\sigma_2\sigma_5 + 5\sigma_3\sigma_5 + \sigma_2\sigma_6 + \sigma_3\sigma_6 + 2\sigma_4\sigma_6 + 3\sigma_5\sigma_6, \\
& 5\sigma_1^2 + 3\sigma_2\sigma_1 + 3\sigma_4\sigma_1 + 3\sigma_5\sigma_1 + 3\sigma_6\sigma_1 + 4\sigma_2^2 + 3\sigma_3^2 + 3\sigma_4^2 + 5\sigma_5^2 + \sigma_6^2 + 5\sigma_2\sigma_4 \\
& + \sigma_3\sigma_4 + 3\sigma_2\sigma_5 + 4\sigma_3\sigma_5 + 5\sigma_4\sigma_5 + 3\sigma_2\sigma_6 + 3\sigma_3\sigma_6 + 2\sigma_4\sigma_6 + 2\sigma_5\sigma_6, \\
& 5\sigma_1^2 + \sigma_2\sigma_1 + 5\sigma_3\sigma_1 + \sigma_4\sigma_1 + 2\sigma_5\sigma_1 + \sigma_6\sigma_1 + 4\sigma_2^2 + 5\sigma_5^2 + 3\sigma_6^2 + \sigma_2\sigma_3 \\
& + 4\sigma_2\sigma_4 + 2\sigma_3\sigma_4 + 3\sigma_2\sigma_5 + 5\sigma_3\sigma_5 + 5\sigma_2\sigma_6 + 3\sigma_3\sigma_6 + 4\sigma_4\sigma_6, \\
& 2\sigma_1^2 + 3\sigma_2\sigma_1 + \sigma_3\sigma_1 + 3\sigma_4\sigma_1 + \sigma_5\sigma_1 + 3\sigma_6\sigma_1 + 4\sigma_2^2 + 5\sigma_3^2 + \sigma_4^2 + 5\sigma_5^2 + 5\sigma_6^2 \\
& + 3\sigma_2\sigma_3 + 4\sigma_2\sigma_4 + 2\sigma_3\sigma_4 + 5\sigma_2\sigma_5 + 3\sigma_3\sigma_5 + 5\sigma_4\sigma_5 + 2\sigma_3\sigma_6 + 4\sigma_4\sigma_6 + 3\sigma_5\sigma_6, \\
& 5\sigma_1^2 + 5\sigma_3\sigma_1 + 4\sigma_4\sigma_1 + 3\sigma_5\sigma_1 + 3\sigma_6\sigma_1 + 5\sigma_2^2 + 3\sigma_3^2 + 2\sigma_4^2 + 4\sigma_5^2 + \sigma_6^2 + 3\sigma_2\sigma_3 \\
& + 5\sigma_2\sigma_4 + \sigma_3\sigma_4 + 3\sigma_2\sigma_5 + 4\sigma_3\sigma_5 + 4\sigma_4\sigma_5 + 4\sigma_2\sigma_6 + 5\sigma_3\sigma_6 + 3\sigma_4\sigma_6 + 5\sigma_5\sigma_6, \\
& 3\sigma_2^2 + 3\sigma_3\sigma_2 + 2\sigma_4\sigma_2 + 5\sigma_5\sigma_2 + 5\sigma_3^2 + 2\sigma_4^2 + 5\sigma_5^2 + \sigma_6^2 + 5\sigma_1\sigma_3 + 5\sigma_1\sigma_4 \\
& + 5\sigma_3\sigma_4 + 5\sigma_1\sigma_5 + 3\sigma_3\sigma_5 + 5\sigma_4\sigma_5 + 4\sigma_1\sigma_6 + 3\sigma_3\sigma_6 + \sigma_4\sigma_6 + 2\sigma_5\sigma_6 \rangle.
\end{aligned}$$

A.2 Inhomogeneous ideals

We have also checked Conjecture 4.2 against non-homogeneous ideals. This case, however, is much more time-consuming to work out and we only present four simple examples below that can be easily processed in a short period of time. The function $g(\sigma_i)$

Table 2: Non-homogeneous ideals

Ideal	Numerical	Residue from the ansatz
$\mathcal{I}_3^{2,3}$	-37.9119	$-\frac{859352384}{22667121}$
$\mathcal{I}_3^{2,4}$	-0.249991	$-\frac{1}{4}$
$\mathcal{I}_3^{3,4}$	0.837891	$\frac{13121}{15625}$
$\mathcal{I}_4^{2,3}$	0.102539	$\frac{167007145709}{1630641375000}$

remains the same as in the homogeneous case and the residues are listed in Table 2.

$$\begin{aligned}
\mathcal{I}_3^{2,3} &= \langle 2\sigma_1^3 + \sigma_3\sigma_1^2 + \sigma_1^2 + 2\sigma_2\sigma_1 + \sigma_3\sigma_1 + \sigma_2^2 + 2\sigma_2\sigma_3^2 + 2\sigma_3^2 + 2\sigma_2^2\sigma_3 + 2\sigma_2\sigma_3, \\
&\quad 2\sigma_1^3 + 2\sigma_2\sigma_1^2 + 2\sigma_3\sigma_1^2 + \sigma_2^2\sigma_1 + \sigma_3^2\sigma_1 + 2\sigma_2^2 + \sigma_3^2 + \sigma_2\sigma_3, \\
&\quad 2\sigma_1^3 + \sigma_2^2\sigma_1 + 2\sigma_3^2\sigma_1 + 2\sigma_2\sigma_1 + \sigma_2\sigma_3\sigma_1 + \sigma_2^2 + \sigma_3^3 + \sigma_2^2 + \sigma_2^2\sigma_3 + 2\sigma_2\sigma_3 \rangle, \\
\mathcal{I}_3^{2,4} &= \langle 2\sigma_1^4 + \sigma_1^2 + \sigma_3^3\sigma_1 + 2\sigma_3^4 + 2\sigma_3^3, \sigma_1^4 + \sigma_1^3 + 2\sigma_2\sigma_1 + 2\sigma_3^4 + \sigma_2^2\sigma_3^2 + 2\sigma_3^2, \\
&\quad 2\sigma_1^4 + \sigma_2\sigma_1^2 + 2\sigma_3\sigma_1 + \sigma_2\sigma_3^3 + 2\sigma_2^2 \rangle, \\
\mathcal{I}_3^{3,4} &= \langle 3\sigma_1^3 + \sigma_2\sigma_3\sigma_1 + 3\sigma_2^2\sigma_3^2 + 3\sigma_2\sigma_3^2, \sigma_1^4 + 5\sigma_2\sigma_1^2 + 5\sigma_2^3 + \sigma_2\sigma_3^3, 3\sigma_2\sigma_3^3 + 5\sigma_3^3 \rangle, \\
\mathcal{I}_4^{2,3} &= \langle 2\sigma_1^2 + 5\sigma_2\sigma_1 + 2\sigma_3\sigma_4^2, 2\sigma_2^2 + 3\sigma_4^2\sigma_2 + 3\sigma_1\sigma_4, 3\sigma_3^2 + \sigma_1\sigma_3 + 2\sigma_1\sigma_4\sigma_3, 2\sigma_1\sigma_4 + 2\sigma_2\sigma_3\sigma_4 + 5\sigma_4\sigma_4 \rangle.
\end{aligned}$$

B Standard method for dealing with poles at infinity

Here we present a standard method of calculating the residue at infinity for the following differential form

$$\Omega = \left(\frac{\sigma_2\sigma_3^2}{h_1h_2h_3} - \frac{\sigma_1\sigma_2\sigma_3}{h_1h_2h_3} \right) d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3 \quad (\text{B.1})$$

where h_i 's are the scattering equations in the gauge $\sigma_4 = 1$

$$h_1 = l_4 + l_1\sigma_1 + l_2\sigma_2 + l_3\sigma_3, \quad (\text{B.2})$$

$$h_2 = -l_{14}\sigma_1 - l_{24}\sigma_2 - l_{34}\sigma_3 - l_{12}\sigma_1\sigma_2 - l_{23}\sigma_2\sigma_3 - l_{13}\sigma_1\sigma_3, \quad (\text{B.3})$$

$$h_3 = l_{124}\sigma_1\sigma_2 + l_{234}\sigma_2\sigma_3 + l_{134}\sigma_1\sigma_3 + l_{123}\sigma_1\sigma_2\sigma_3. \quad (\text{B.4})$$

Let us first recall how to calculate the residue at infinity in the single variable case. In that case we have, after changing variable $z \rightarrow 1/\xi$,

$$\oint_{z=\infty} f(z)dz = \oint_{\xi=0} f\left(\frac{1}{\xi}\right) d\left(\frac{1}{\xi}\right). \quad (\text{B.5})$$

In the language of differential geometry we are actually considering the complex plane as one of two standard patches U_0 and U_1 covering \mathbb{CP}^1 , i.e $U_0 = \{[z_0, z_1] | z_0 \neq 0\}$ and $U_1 = \{[z_0, z_1] | z_1 \neq 0\}$, where $z_{0,1}$ are homogenous coordinates for \mathbb{CP}^1 . Then the point of infinity is just the single point of \mathbb{CP}^1 that is missing in U_0 , i.e $\infty = [0, 1] \in \mathbb{CP}^1$. It is not on U_0 but on U_1 , and the change of variable $z \rightarrow 1/\xi$ is just the coordinate transformation when we go from patch U_0 to U_1 where we can calculate the residue.

Similarly we can define the residue at infinity for the multivariable case. But in this case there is actually a hypersurface, instead of a single point, located at infinity. This can be seen as follows. To be specific we discuss the calculation in the form of (B.1). Thus we are considering the form Ω on \mathbb{C}^3 . Firstly we need to embed \mathbb{C}^3 into a compact manifold to be able to use the global residue theorem. \mathbb{CP}^3 is a natural choice. The original \mathbb{C}^3 can be identified with one of the standard patches covering \mathbb{CP}^3 , say $U_0 = \{[z_0, z_1, z_2, z_3] | z_0 \neq 0\}$. In that sense, what is now at infinity is the hypersurface $\{[z_0, z_1, z_2, z_3] | z_0 = 0\}$. Eq. (B.1) is now interpreted as the local expression on U_0 of a form on \mathbb{CP}^3 , i.e in terms of the homogenous coordinates,

$$\Omega = \frac{(z_3/z_0 - z_1/z_0)(z_2/z_0)(z_3/z_0)}{h_1 h_2 h_3} d\left(\frac{z_1}{z_0}\right) \wedge d\left(\frac{z_2}{z_0}\right) \wedge d\left(\frac{z_3}{z_0}\right). \quad (\text{B.6})$$

And h_i 's are expressed in terms of homogenous coordinates as well

$$h_1 = (l_4 z_0 + l_1 z_1 + l_2 z_2 + l_3 z_3) z_0^{-1} =: \tilde{h}_1 z_0^{-1}, \quad (\text{B.7})$$

$$h_2 = (-l_{14} z_0 z_1 - l_{24} z_0 z_2 - l_{34} z_0 z_3 - l_{12} z_1 z_2 - l_{23} z_2 z_3 - l_{13} z_1 z_3) z_0^{-2} =: \tilde{h}_2 z_0^{-2}, \quad (\text{B.8})$$

$$h_3 = (l_{124} z_0 z_1 z_2 + l_{234} z_0 z_2 z_3 + l_{134} z_0 z_1 z_3 + l_{123} z_1 z_2 z_3) z_0^{-3} =: \tilde{h}_3 z_0^{-3}. \quad (\text{B.9})$$

Furthermore the \tilde{h}_i , $i = 1, 2, 3$ as defined above are homogenous functions of degrees 1, 2, 3 respectively. It follows that

$$\begin{aligned} \Omega &= \frac{(z_3 - z_1) z_2 z_3}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_3 z_0} \\ &\times (z_0 dz_1 \wedge dz_2 \wedge dz_3 - z_1 dz_0 \wedge dz_2 \wedge dz_3 - z_2 dz_1 \wedge dz_0 \wedge dz_3 - z_3 dz_1 \wedge dz_2 \wedge dz_0), \end{aligned} \quad (\text{B.10})$$

from which Ω is singular on $\tilde{h}_1 = 0$, $\tilde{h}_2 = 0$, $\tilde{h}_3 = 0$ and $z_0 = 0$. To define residues on \mathbb{CP}^3 we need to regard the denominator $\tilde{h}_1 \tilde{h}_2 \tilde{h}_3 z_0$ as the product of three divisors.

There are multiple ways to do so, each of which leads to an equation of global residue theorem. One choice, however, is particularly simple, i.e

$$\begin{aligned} D_1 &= \tilde{h}_1 z_0, \\ D_2 &= \tilde{h}_2, \\ D_3 &= \tilde{h}_3. \end{aligned} \tag{B.11}$$

With this choice the global residue theorem reads, with S denoting the set of common zeros of these divisors,

$$0 = \sum_{p \in S} \text{Res}_{\{D_1, D_2, D_3\}, p} \Omega = \sum_{p \in S \cap U_0} \text{Res}_{\{D_1, D_2, D_3\}, p} \Omega + \sum_{p \in S_\infty} \text{Res}_{\{D_1, D_2, D_3\}, p} \Omega \tag{B.12}$$

The first term is what we originally want to calculate, i.e the sum of all the residues *not* at infinity. The second term contains contributions from points at infinity. S_∞ denotes the set of poles at infinity, and is easily seen to be

$$S_\infty = \{p_1 := [0, 1, 0, 0], p_2 := [0, 0, 1, 0], p_3 := [0, 0, 0, 1]\} \tag{B.13}$$

whose elements are on U_1 , U_2 and U_3 respectively. We now go on to each of these three patches to compute residues there.

On U_1 we set $z_1 = 1$ and the form Ω is

$$U_1 : \quad \Omega|_{U_1} = - \frac{(z_3 - 1)z_2 z_3}{z_0 \cdot \left(\tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \right) \Big|_{z_1=1}} dz_0 \wedge dz_2 \wedge dz_3. \tag{B.14}$$

The divisor choice (B.11) becomes⁹

$$U_1 : \quad D_1 = z_0 \tilde{h}_1(z_1 = 1), \quad D_2 = \tilde{h}_2(z_1 = 1), \quad D_3 = \tilde{h}_3(z_1 = 1). \tag{B.15}$$

Thus

$$\oint_{p_1} \Omega|_{U_1} = \oint_{z_0=z_2=z_3=0} \left[- \frac{(z_3 - 1)z_2 z_3}{z_0 \cdot \left(\tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \right) \Big|_{z_1=1}} dz_0 \wedge dz_2 \wedge dz_3 \right] = 0 \tag{B.16}$$

and the residue at p_1 vanishes.

⁹Since $\tilde{h}_1(z_1 = 1, z_1 = z_2 = z_3 = 0) \neq 0$ the divisors are actually $\{z_0, \tilde{h}_2, \tilde{h}_3\} \Big|_{z_0=1}$.

On U_2 we set $z_2 = 1$ in Ω and choose the divisors in the same fashion, and the residue at p_2 also vanishes

$$\oint_{p_2} \Omega|_{U_2} = \oint_{z_0=z_1=z_3=0} \left[-\frac{(z_3 - z_1)z_3}{z_0 \cdot \left(\tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \right) \Big|_{z_2=1}} dz_0 \wedge dz_2 \wedge dz_3 \right] = 0. \quad (\text{B.17})$$

The residue at p_3 is nontrivial. On U_3 we set $z_3 = 1$ and have

$$\begin{aligned} \oint_{p_3} \Omega|_{U_3} &= \oint_{z_0=z_1=z_3=0} \left[-\frac{(1 - z_1)z_2}{z_0 \cdot \left(\tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \right) \Big|_{z_2=1}} dz_1 \wedge dz_2 \wedge dz_0 \right] \\ &= -(l_{123} \cdot l_{23} \cdot l_3)^{-1}. \end{aligned} \quad (\text{B.18})$$

Combining (B.16), (B.17) and (B.18) we get

$$\sum_{p \in S_\infty} \text{Res}_{\{D_1, D_2, D_3\}, p} \Omega = -(l_{123} \cdot l_{23} \cdot l_3)^{-1}. \quad (\text{B.19})$$

Using momentum conservation, this is equal to

$$\frac{1}{l \cdot k_3 (-k_{23} + l \cdot k_2 + l \cdot k_3) (l \cdot k_4)}. \quad (\text{B.20})$$

By (B.12) we then immediately get the value of $\sum_{p \in S \cap U_0} \text{Res}_{\{D_1, D_2, D_3\}, p} \Omega$, which agrees with (5.23) found using our ansatz.

C Obtaining the residue by inspection

As shown in Section 5, we can evaluate the CHY-form conveniently using our ansatz. This method in fact applies to all kinds of meromorphic differential forms, and the intrinsic structure of the scattering equations has not been fully explored in calculations. From the discussion on residues at infinity in section B, it seems the particular form of scattering equations in fact greatly simplifies the evaluation process of the standard method. A natural question is then whether that could also help simplify the calculation using the ansatz method. As an example, in this section we discuss the following term from the 4-point one-loop super Yang-Mills amplitude

$$\oint_{h_1=h_2=h_3=0} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1 h_2 h_3} \mathcal{R}_4^{(1)} = - \oint_{\tilde{h}_0=\dots=\tilde{h}_3=0} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3 \wedge d\sigma_0}{\tilde{h}_0 \tilde{h}_1 \tilde{h}_2 \tilde{h}_3} \frac{\sigma_3(\sigma_2 - 1)}{(\sigma_0 - 1)}. \quad (\text{C.1})$$

Many equations from the local duality theorem contain only one of the a_{ijkl} 's. Such equations simply lead to the vanishing of those coefficients. The surviving equations from the local duality theorem are

$$\begin{pmatrix} 0 & -l_{34} & 0 & -l_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & l_4 & 0 & l_2 & 0 & 0 & 0 & 0 \\ l_3 & 0 & l_2 & 0 & 0 & 0 & 0 & 0 & 3l_4 \\ l_4 & l_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_4 & 0 & l_2 & 0 & 0 & 0 & 3l_3 & 0 \\ 0 & 0 & 0 & l_3 & 0 & l_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_4 & l_3 & 3l_2 & 0 & 0 \\ -l_{34} & 0 & -l_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -l_{24} & -l_{23} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{0012} \\ a_{0021} \\ a_{0102} \\ a_{0120} \\ a_{0201} \\ a_{0210} \\ a_{0300} \\ a_{0030} \\ a_{0003} \end{pmatrix} = 0. \quad (\text{C.2})$$

The equation for the intersection number is

$$\begin{aligned} & (-2l_1l_4l_{12}, 2l_1l_3l_{12}, 2l_1l_4l_{13}, -2l_1l_3l_{14}, -2l_1l_2l_{13}, 2l_1l_2l_{14}) \\ & \cdot (a_{0012}, a_{0021}, a_{0102}, a_{0120}, a_{0201}, a_{0210})^T = 6. \end{aligned} \quad (\text{C.3})$$

From these equations it is possible to read off $a_{0012}, a_{0021}, a_{0102}, a_{0120}, a_{0201}, a_{0210}$ by inspection

$$\begin{aligned} & (a_{0012}, a_{0021}, a_{0102}, a_{0120}, a_{0201}, a_{0210}) \\ & = \left(\frac{1}{2l_1l_4l_{34}}, \frac{1}{-2l_1l_3l_{34}}, \frac{1}{-2l_1l_4l_{24}}, \frac{1}{2l_1l_3l_{23}}, \frac{1}{2l_1l_2l_{24}}, \frac{1}{-2l_1l_2l_{23}} \right). \end{aligned} \quad (\text{C.4})$$

Namely the value of each a is simply the reciprocal of the coefficient in front of it. We will see that this pattern also appears in later calculations for $\mathcal{R}_4^{(2)}, \mathcal{R}_4^{(3)}$ and $\mathcal{R}_4^{(4)}$. Now this knowledge is already enough for us to evaluate (C.1) using our conjecture because

$$\text{Res}_{\{(\tilde{h}_1), (\tilde{h}_2), (\tilde{h}_3), (\tilde{h}_0)\}} \left(\frac{\sigma_3(\sigma_2 - 1)}{(\sigma_0 - 1)} \right) = \mathbb{D} \left(\frac{\sigma_3\sigma_2 - \sigma_3}{\sigma_0 - 1} \right) = \mathbb{D} \left(\frac{-\sigma_3}{\sigma_0 - 1} \right) \quad (\text{C.5})$$

which involves only a_{0012} and therefore

$$\text{Res}_{\{(h_1), (h_2), (h_3)\}} (\mathcal{R}_4^{(1)}) = 2a_{0012} = \frac{1}{l_1l_4l_{34}}.$$

The other three terms involving $\mathcal{R}_4^{(2)}, \mathcal{R}_4^{(3)}, \mathcal{R}_4^{(4)}$ can be calculated similarly. In evaluating the residue for $\mathcal{R}_4^{(2)}$, the intersection number equation is

$$2l_1l_2l_{13}a_{2010} - 2l_2l_3l_{13}a_{1020} - 2l_2l_4l_{34}a_{0012} + 2l_2l_3l_{34}a_{0021} + 2l_2l_4l_{14}a_{1002} - 2l_1l_2l_{14}a_{2001} = 6.$$

As mentioned above, the a 's are again reciprocals of their respective coefficients

$$\begin{aligned} & (a_{2010}, a_{1020}, a_{0012}, a_{0021}, a_{1002}, a_{2001}) \\ &= \left(\frac{1}{2l_1l_2l_{13}}, \frac{1}{-2l_2l_3l_{13}}, \frac{1}{-2l_2l_4l_{34}}, \frac{1}{2l_2l_3l_{34}}, \frac{1}{2l_2l_4l_{14}}, \frac{1}{-2l_1l_2l_{14}} \right) \end{aligned} \quad (\text{C.6})$$

which at the same time solve other equations from local duality theorem. Thus

$$\text{Res}_{\{(h_1), (h_2), (h_3)\}}(\mathcal{R}_4^{(2)}) = -2a_{2001} = \frac{1}{l_1l_2l_{14}}.$$

For $\mathcal{R}_4^{(3)}$, The intersection equation is

$$-2l_2l_3l_{24}a_{0,2,0,1} + 2l_4l_3l_{24}a_{0102} - 2l_1l_3l_{12}a_{2100} + 2l_2l_3l_{12}a_{1200} + 2l_1l_3l_{14}a_{2001} - 2l_3l_4l_{14}a_{1002} = 6,$$

from whose coefficients we find the solution to be

$$\begin{aligned} & (a_{0201}, a_{0102}, a_{2100}, a_{1200}, a_{2001}, a_{1002}) \\ &= \left(\frac{1}{-2l_2l_3l_{24}}, \frac{1}{2l_4l_3l_{24}}, \frac{1}{-2l_1l_3l_{12}}, \frac{1}{2l_2l_3l_{12}}, \frac{1}{2l_1l_3l_{14}}, \frac{1}{-2l_3l_4l_{14}} \right). \end{aligned} \quad (\text{C.7})$$

So we have

$$\text{Res}_{\{(h_1), (h_2), (h_3)\}}(\mathcal{R}_4^{(3)}) = 2a_{1200} = \frac{1}{-l_2l_3l_{12}}.$$

For $\mathcal{R}_4^{(4)}$, the intersection equation is

$$2l_4l_3l_{13}a_{1020} - 2l_1l_4l_{13}a_{2010} - 2l_1l_4l_{12}a_{2100} - 2l_2l_4l_{12}a_{1200} + 2l_4l_3l_{23}a_{0120} + 2l_2l_4l_{23}a_{0210} = 6.$$

And the solution is

$$\begin{aligned} & (a_{1020}, a_{2010}, a_{2100}, a_{1200}, a_{0120}, a_{0210}) \\ &= \left(\frac{1}{2l_4l_3l_{13}}, \frac{1}{-2l_1l_4l_{13}}, \frac{1}{-2l_1l_4l_{12}}, \frac{1}{-2l_2l_4l_{12}}, \frac{1}{-2l_3l_4l_{23}}, \frac{1}{2l_2l_4l_{23}} \right). \end{aligned} \quad (\text{C.8})$$

Thus

$$\text{Res}_{\{(h_1), (h_2), (h_3)\}}(\mathcal{R}_4^{(4)}) = -2a_{0120} = \frac{1}{l_3l_4l_{23}}.$$

We expect such ease of finding the ansatz solution to appear also for higher point one-loop amplitudes, and will discuss it in future projects.

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References

- [1] F. Cachazo, S. He, and E. Y. Yuan, *Scattering of Massless Particles in Arbitrary Dimensions*, *Phys. Rev. Lett.* **113** (2014), no. 17 171601, [[arXiv:1307.2199](#)].
- [2] F. Cachazo, S. He, and E. Y. Yuan, *Scattering of Massless Particles: Scalars, Gluons and Gravitons*, *JHEP* **07** (2014) 033, [[arXiv:1309.0885](#)].
- [3] F. Cachazo, S. He, and E. Y. Yuan, *Einstein-Yang-Mills Scattering Amplitudes From Scattering Equations*, *JHEP* **01** (2015) 121, [[arXiv:1409.8256](#)].
- [4] H. Kawai, D. C. Lewellen, and S. H. H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys.* **B269** (1986) 1–23.
- [5] F. Cachazo, S. He, and E. Y. Yuan, *Scattering equations and Kawai-Lewellen-Tye orthogonality*, *Phys. Rev.* **D90** (2014), no. 6 065001, [[arXiv:1306.6575](#)].
- [6] R. Kleiss and H. Kuijf, *Multi - Gluon Cross-sections and Five Jet Production at Hadron Colliders*, *Nucl. Phys.* **B312** (1989) 616–644.
- [7] Z. Bern, J. J. M. Carrasco, and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev.* **D78** (2008) 085011, [[arXiv:0805.3993](#)].
- [8] F. Cachazo, S. He, and E. Y. Yuan, *Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM*, *JHEP* **07** (2015) 149, [[arXiv:1412.3479](#)].
- [9] R. Roiban, M. Spradlin, and A. Volovich, *On the tree level S matrix of Yang-Mills theory*, *Phys. Rev.* **D70** (2004) 026009, [[hep-th/0403190](#)].
- [10] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189–258, [[hep-th/0312171](#)].
- [11] F. Cachazo, L. Mason, and D. Skinner, *Gravity in Twistor Space and its Grassmannian Formulation*, *SIGMA* **10** (2014) 051, [[arXiv:1207.4712](#)].
- [12] D. Skinner, *Twistor Strings for N=8 Supergravity*, [arXiv:1301.0868](#).

- [13] F. Cachazo, S. He, and E. Y. Yuan, *Scattering in Three Dimensions from Rational Maps*, *JHEP* **10** (2013) 141, [[arXiv:1306.2962](#)].
- [14] F. Cachazo and Y. Geyer, *A 'Twistor String' Inspired Formula For Tree-Level Scattering Amplitudes in $N=8$ SUGRA*, [arXiv:1206.6511](#).
- [15] F. Cachazo and D. Skinner, *Gravity from Rational Curves in Twistor Space*, *Phys. Rev. Lett.* **110** (2013), no. 16 161301, [[arXiv:1207.0741](#)].
- [16] L. Dolan and P. Goddard, *The Polynomial Form of the Scattering Equations*, *JHEP* **07** (2014) 029, [[arXiv:1402.7374](#)].
- [17] Y.-H. He, C. Matti, and C. Sun, *The Scattering Variety*, *JHEP* **10** (2014) 135, [[arXiv:1403.6833](#)].
- [18] C. Kalousios, *Massless scattering at special kinematics as Jacobi polynomials*, *J. Phys.* **A47** (2014) 215402, [[arXiv:1312.7743](#)].
- [19] S. Weinzierl, *On the solutions of the scattering equations*, *JHEP* **04** (2014) 092, [[arXiv:1402.2516](#)].
- [20] C. S. Lam, *Permutation Symmetry of the Scattering Equations*, *Phys. Rev.* **D91** (2015), no. 4 045019, [[arXiv:1410.8184](#)].
- [21] Y.-j. Du, F. Teng, and Y.-s. Wu, *CHY formula and MHV amplitudes*, *JHEP* **05** (2016) 086, [[arXiv:1603.08158](#)].
- [22] C. Kalousios, *Scattering equations, generating functions and all massless five point tree amplitudes*, *JHEP* **05** (2015) 054, [[arXiv:1502.07711](#)].
- [23] C. Cardona and C. Kalousios, *Comments on the evaluation of massless scattering*, *JHEP* **01** (2016) 178, [[arXiv:1509.08908](#)].
- [24] C. Cardona and C. Kalousios, *Elimination and recursions in the scattering equations*, *Phys. Lett.* **B756** (2016) 180–187, [[arXiv:1511.05915](#)].
- [25] L. Dolan and P. Goddard, *General Solution of the Scattering Equations*, [arXiv:1511.09441](#).
- [26] R. Huang, J. Rao, B. Feng, and Y.-H. He, *An Algebraic Approach to the Scattering Equations*, *JHEP* **12** (2015) 056, [[arXiv:1509.04483](#)].
- [27] M. Sogaard and Y. Zhang, *Scattering Equations and Global Duality of Residues*, *Phys. Rev.* **D93** (2016), no. 10 105009, [[arXiv:1509.08897](#)].
- [28] J. Bosma, M. Słagaard, and Y. Zhang, *The Polynomial Form of the Scattering Equations is an H -Basis*, *Phys. Rev.* **D94** (2016), no. 4 041701, [[arXiv:1605.08431](#)].
- [29] M. Zlotnikov, *Polynomial reduction and evaluation of tree- and loop-level CHY amplitudes*, *JHEP* **08** (2016) 143, [[arXiv:1605.08758](#)].

- [30] F. Cachazo and H. Gomez, *Computation of Contour Integrals on $\mathcal{M}_{0,n}$* , *JHEP* **04** (2016) 108, [[arXiv:1505.03571](#)].
- [31] H. Gomez, *Λ scattering equations*, *JHEP* **06** (2016) 101, [[arXiv:1604.05373](#)].
- [32] C. Cardona and H. Gomez, *Elliptic scattering equations*, *JHEP* **06** (2016) 094, [[arXiv:1605.01446](#)].
- [33] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily, and P. H. Damgaard, *Integration Rules for Scattering Equations*, *JHEP* **09** (2015) 129, [[arXiv:1506.06137](#)].
- [34] C. S. Lam and Y.-P. Yao, *Role of $M\check{Z}$ bius constants and scattering functions in Cachazo-He-Yuan scalar amplitudes*, *Phys. Rev.* **D93** (2016), no. 10 105004, [[arXiv:1512.05387](#)].
- [35] C. S. Lam and Y.-P. Yao, *Evaluation of the Cachazo-He-Yuan gauge amplitude*, *Phys. Rev.* **D93** (2016), no. 10 105008, [[arXiv:1602.06419](#)].
- [36] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily, P. H. Damgaard, and B. Feng, *Integration Rules for Loop Scattering Equations*, *JHEP* **11** (2015) 080, [[arXiv:1508.03627](#)].
- [37] C. R. Mafra, *Berends-Giele recursion for double-color-ordered amplitudes*, *JHEP* **07** (2016) 080, [[arXiv:1603.09731](#)].
- [38] R. Huang, B. Feng, M.-x. Luo, and C.-J. Zhu, *Feynman Rules of Higher-order Poles in CHY Construction*, *JHEP* **06** (2016) 013, [[arXiv:1604.07314](#)].
- [39] N. E. J. Bjerrum-Bohr, J. L. Bourjaily, P. H. Damgaard, and B. Feng, *Analytic Representations of Yang-Mills Amplitudes*, [arXiv:1605.06501](#).
- [40] C. Cardona, B. Feng, H. Gomez, and R. Huang, *Cross-ratio Identities and Higher-order Poles of CHY-integrand*, [arXiv:1606.00670](#).
- [41] F. Cachazo, S. He, and E. Y. Yuan, *One-Loop Corrections from Higher Dimensional Tree Amplitudes*, *JHEP* **08** (2016) 008, [[arXiv:1512.05001](#)].
- [42] B. Feng, *CHY-construction of Planar Loop Integrands of Cubic Scalar Theory*, *JHEP* **05** (2016) 061, [[arXiv:1601.05864](#)].
- [43] N. Berkovits, *Infinite Tension Limit of the Pure Spinor Superstring*, *JHEP* **03** (2014) 017, [[arXiv:1311.4156](#)].
- [44] L. Mason and D. Skinner, *Ambitwistor strings and the scattering equations*, *JHEP* **07** (2014) 048, [[arXiv:1311.2564](#)].
- [45] Y. Geyer, A. E. Lipstein, and L. J. Mason, *Ambitwistor Strings in Four Dimensions*, *Phys. Rev. Lett.* **113** (2014), no. 8 081602, [[arXiv:1404.6219](#)].
- [46] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, *Loop Integrands for Scattering Amplitudes from the Riemann Sphere*, *Phys. Rev. Lett.* **115** (2015), no. 12 121603,

- [arXiv:1507.00321].
- [47] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, *One-loop amplitudes on the Riemann sphere*, *JHEP* **03** (2016) 114, [arXiv:1511.06315].
 - [48] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, *Two-Loop Scattering Amplitudes from the Riemann Sphere*, arXiv:1607.08887.
 - [49] D. A. Cox, J. Little, and D. O’shea, *Using algebraic geometry*, vol. 185. Springer Science & Business Media, 2006.
 - [50] R. Hartshorne, *Algebraic geometry*, vol. 52. Springer Science & Business Media, 2013.
 - [51] P. Griffiths and J. Harris, *Principles of algebraic geometry*. John Wiley & Sons, 2014.
 - [52] D. J. Gross and J. L. Manes, *The High-energy Behavior of Open String Scattering*, *Nucl. Phys.* **B326** (1989) 73–107.
 - [53] Z. Bern, L. J. Dixon, and V. A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev.* **D72** (2005) 085001, [hep-th/0505205].
 - [54] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, and J. Trnka, *Singularity Structure of Maximally Supersymmetric Scattering Amplitudes*, *Phys. Rev. Lett.* **113** (2014), no. 26 261603, [arXiv:1410.0354].
 - [55] S. Franco, D. Galloni, B. Penante, and C. Wen, *Non-Planar On-Shell Diagrams*, *JHEP* **06** (2015) 199, [arXiv:1502.02034].
 - [56] B. Chen, G. Chen, Y.-K. E. Cheung, R. Xie, and Y. Xin, *Top-forms of Leading Singularities in Nonplanar Multi-loop Amplitudes*, arXiv:1506.02880.
 - [57] B. Chen, G. Chen, Y.-K. E. Cheung, R. Xie, and Y. Xin, *Top-forms of Leading Singularities in Nonplanar Multi-loop Amplitudes*, arXiv:1507.03214.
 - [58] J. L. Bourjaily, S. Franco, D. Galloni, and C. Wen, *Stratifying On-Shell Cluster Varieties: the Geometry of Non-Planar On-Shell Diagrams*, arXiv:1607.01781.
 - [59] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily, S. Caron-Huot, P. H. Damgaard, and B. Feng, *New Representations of the Perturbative S-Matrix*, *Phys. Rev. Lett.* **116** (2016), no. 6 061601, [arXiv:1509.02169].
 - [60] R. Huang, Q. Jin, J. Rao, K. Zhou, and B. Feng, *The Q-cut Representation of One-loop Integrands and Unitarity Cut Method*, *JHEP* **03** (2016) 057, [arXiv:1512.02860].
 - [61] W. Ebeling, *Functions of several complex variables and their singularities*, vol. 83. American Mathematical Soc., 2007.
 - [62] G.-M. Greuel, C. Lossen, and E. I. Shustin, *Introduction to singularities and deformations*. Springer Science & Business Media, 2007.